Using Substitution
Homogeneous and Bernoulli Equations

Sometimes differential equations may not appear to be in a solvable form. However, if we make an appropriate substitution, often the equations can be forced into forms which we can solve, much like the use of \( u \) substitution for integration. We must be careful to make the appropriate substitution. Two particular forms of equations lend themselves naturally to substitution.

**Homogeneous Equations** A function \( F(x, y) \) is said to be *homogeneous* if for some \( t \neq 0 \)

\[
F(tx, ty) = F(x, y).
\]

That is to say that a function is homogeneous if replacing the variables by a scalar multiple does not change the equation. Please note that the term homogeneous is used for two different concepts in differential equations.

**Examples**

\[
F(x, y) = \frac{-3y}{3x - 7y} \quad \text{is homogeneous since}
\]

\[
F(tx, ty) = \frac{-3ty}{3tx - 7ty} = \frac{-3ty}{t(3x - 7y)} = \frac{-3y}{3x - 7y} = F(x, y).
\]

\[
F(x, y) = \frac{xy^2 + x^3 \cos \left( \frac{2x}{3y} \right)}{5y^2 x + y^3} \quad \text{is homogeneous since}
\]

\[
F(tx, ty) = \frac{tx(ty)^2 + (tx)^3 \cos \left( \frac{2tx}{3ty} \right)}{5(ty)^2tx + (ty)^3} = \frac{t^3xy^2 + t^3x^3 \cos \left( \frac{2tx}{3ty} \right)}{5t^2y^2tx + t^3y^3} = \frac{t^3(xy^2 + x^3 \cos \left( \frac{2tx}{3ty} \right))}{t^3(5y^2 x + y^3)}
\]

\[
= \frac{xy^2 + x^3 \cos \left( \frac{2x}{3y} \right)}{5y^2 x + y^3} = F(x, y)
\]

We say that a differential equation is homogeneous if it is of the form \( \frac{dy}{dx} = F(x, y) \) for a homogeneous function \( F(x, y) \). If this is the case, then we can make the substitution \( y = ux \). After using this substitution, the equation can be solved as a *separable* differential equation. After solving, we again use the substitution \( y = ux \) to express the answer as a function of \( x \) and \( y \).
Example

1. \( \frac{dy}{dx} = \frac{-3y}{3x - 7y} \)

We have already seen that the function above is homogeneous from the previous examples. As a result, this is a homogeneous differential equation. We will substitute \( y = ux \). By the product rule,

\[
\frac{dy}{dx} = x \frac{du}{dx} + u.
\]

Making these substitutions we obtain

\[
\frac{du}{dx} + u = \frac{-3ux}{3x - 7ux}.
\]

Now this equation must be separated.

\[
x \frac{du}{dx} + u = \frac{-3ux}{x(3 - 7u)} \implies \frac{du}{dx} + \frac{u}{3 - 7u} = \frac{-3u}{3 - 7u} \implies \frac{du}{dx} = \frac{-6u + 7u^2}{3 - 7u}.
\]

Integrating this we get,

\[
-\frac{1}{2} \ln(-6u + 7u^2) = \ln(x) + C \implies \frac{1}{\sqrt{-6u + 7u^2}} = cx \implies \frac{1}{6u + 7u^2} = cx^2.
\]

Finally we use that \( u = \frac{y}{x} \) to get our implicit solution

\[
-6yx + 7y^2 = cx^2 \implies -6yx + 7y^2 = c.
\]

**Bernoulli Equations** We say that a differential equation is a *Bernoulli Equation* if it takes one of the forms

\[
y^m \frac{dy}{dx} + p(x)y^{m+1} = q(x), \quad \frac{dy}{dx} + p(x)y = q(x)y^n.
\]

These differential equations almost match the form required to be linear. By making a substitution, both of these types of equations can be made to be linear. Those of the first type require the substitution \( v = y^{m+1} \). Those of the second type require the substitution \( u = y^{1-n} \). Once these substitutions are made, the equation will be *linear* and may be solved accordingly.

**Example**

\[
(a) \quad \frac{dy}{dx} - \frac{y}{3x} = e^x y^4
\]
You can see that this is a Bernoulli equation of the second form. We make the substitution $u = y^{1-4} = y^{-3}$. This gives $\frac{du}{dx} = -3y^{-4} \frac{dy}{dx}$. The equation will be easier to manipulate if we multiply both sides by $y^{-4}$. Our new equation will be $\frac{y^{-4}dy}{dx} - \frac{y^{-3}}{3x} = e^x$.

Making the appropriate substitutions this becomes

$$\frac{-1}{3} \frac{du}{dx} - \frac{u}{3x} = e^x.$$

If we multiply by $-3$ we see that the equation is now linear in $u$ and can be solved:

$$\frac{du}{dx} + \frac{u}{x} = -3e^x \implies ux = \int -3xe^x dx \implies ux = -3(xe^x - e^x + C)$$

$$\implies u = -3(e^x - \frac{e^x}{x} + \frac{C}{x})$$

After undoing the $u$ substitution, we have the solution

$$\frac{1}{y^3} = -3(e^x - \frac{e^x}{x} + \frac{C}{x}) \implies y^3 = \frac{x}{e^x - xe^x + C}$$