Using Substitution
Homogeneous and Bernoulli Equations

BCCC Tutoring Center

Sometimes differential equations may not appear to be in a solvable form. However, if we make an appropriate substitution, often the equations can be forced into forms which we can solve, much like the use of a substitution for integration. We must be careful to make the appropriate substitution. Two particular forms of equations lend themselves naturally to substitution.

**Homogeneous Equations** A function $F(x, y)$ is said to be *homogeneous* if for some $t \neq 0$

\[ F(tx, ty) = F(x, y). \]

That is to say that a function is homogeneous if replacing the variables by a scalar multiple does not change the equation. Please note that the term homogeneous is used for two different concepts in differential equations.

**Examples**

1. $F(x, y) = \frac{-3y}{3x - 7y}$ is homogeneous since

   \[ F(tx, ty) = \frac{-3ty}{3tx - 7ty} = \frac{-3ty}{t(3x - 7y)} = \frac{-3y}{3x - 7y} = F(x, y). \]

2. $F(x, y) = \frac{xy^2 + x^3 \cos\left(\frac{2x}{3y}\right)}{5y^2x + y^3}$ is homogeneous since

   \[ F(tx, ty) = \frac{tx(ty)^2 + (tx)^3 \cos\left(\frac{2tx}{3ty}\right)}{5(ty)^2tx + (ty)^3} = \frac{t^3x y^2 + t^3x^3 \cos\left(\frac{2tx}{3ty}\right)}{5t^2y^2tx + t^3y^3} = \frac{t^3(x y^2 + x^3 \cos\left(\frac{2tx}{3ty}\right))}{t^3(5y^2x + y^3)} = F(x, y) \]

We say that a differential equation is homogeneous if it is of the form $\frac{dy}{dx} = F(x, y)$ for a homogeneous function $F(x, y)$. If this is the case, then we can make the substitution $y = ux$. After using this substitution, the equation can be solved as a separable differential equation. After solving, we again use the substitution $y = ux$ to express the answer as a function of $x$ and $y$. 

Example

1. \[ \frac{dy}{dx} = \frac{-3y}{3x - 7y} \]

We have already seen that the function on the left is homogeneous from the previous examples. As a result, this is a homogeneous differential equation. We will substitute \( y = ux \). By the product rule, \( \frac{dy}{dx} = x \frac{du}{dx} + u \). Making these substitutions we obtain

\[ \frac{du}{dx} + u = \frac{-3ux}{3x - 7ux} \]

Now this equation must be separated.

\[ x \frac{du}{dx} + u = \frac{-3ux}{x(3 - 7u)} \implies \frac{du}{dx} + u = \frac{-3u}{3 - 7u} \implies \frac{du}{dx} = \frac{-6u + 7u^2}{3 - 7u} \]

Integrating this we get,

\[ -\frac{1}{2} \ln(-6u + 7u^2) = \ln(x) + C \implies \frac{1}{\sqrt{-6u + 7u^2}} = cx \implies \frac{1}{-6u + 7u^2} = cx^2. \]

Finally we use that \( u = \frac{y}{x} \) to get our implicit solution \( \frac{x^2}{-6yx + 7y^2} = cx^2 \implies -6yx + 7y^2 = c. \)

Bernoulli Equations We say that a differential equation is a Bernoulli Equation if it takes one of the forms

\[ y^n \frac{dy}{dx} + p(x)y^{n+1} = q(x), \quad \frac{dy}{dx} + p(x)y = q(x)y^n. \]

These differential equations almost match the form required to be linear. By making a substitution, both of these types of equations can be made to be linear. Those of the first type require the substitution \( v = y^{m+1} \). Those of the second type require the substitution \( u = y^{1-n} \). Once these substitutions are made, the equation will be linear and may be solved accordingly.
Example

(a) \( \frac{dy}{dx} - \frac{y}{3x} = e^x y^4 \)

You can see that this is a Bernoulli equation of the second form. We make the substitution \( u = y^{1-4} = y^{-3} \). This gives \( \frac{du}{dx} = -3y^{-4} \frac{dy}{dx} \). The equation will be easier to manipulate if we multiply both sides by \( y^{-4} \). Our new equation will be \( y^{-4} \frac{dy}{dx} - \frac{y^{-3}}{3x} = e^x \).

Making the appropriate substitutions this becomes \( \frac{-1}{3} \frac{du}{dx} - \frac{u}{3x} = e^x \).

If we multiply by \(-3\) we see that the equation is now linear in \( u \) and can be solved:

\[
\frac{du}{dx} + \frac{u}{x} = -3e^x \quad \Rightarrow \quad ux = \int -3e^x \, dx \quad \Rightarrow \quad ux = -3(xe^x - e^x + C)
\]

\[
\Rightarrow \quad u = -3(e^x - \frac{e^x}{x} + \frac{C}{x})
\]

After undoing the \( u \) substitution, we have the solution

\[
\frac{1}{y^3} = -3(e^x - \frac{e^x}{x} + \frac{C}{x}) \quad \Rightarrow \quad y^3 = \frac{x}{e^x - xe^x + C}
\]