MATH 250- Sample Final Exam

1. Solve the DE
$(8xy + 5y^3)dx + (4x^2 + 15xy^2 + 2y)dy = 0$

2. Solve the DE
$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$

3. Solve the DE
$\frac{dy}{dx} - \frac{1}{x}y = xy^3$

4. Solve the DE
$\frac{d^2y}{dx^2} + 36y = sin\,6x$

5. Find the Laplace transform of the functions.
a) $F(t) = 5t^2 + cos\,8t + sin\,10t$
b) $F(t) = e^{-4t}t^2$
c) $F(t) = e^{9t}cos\,7t$

6. Find the inverse transform of the functions.
a) $f(s) = \frac{1}{s^2} + \frac{4}{(s-2)^2}$
b) $f(s) = \frac{3}{s^2 + 4s + 29}$
c) $f(s) = \frac{3\,e^{-4s}}{s^2 + 36}$
d) $f(s) = \frac{s+9}{s^2(s^2+4)}$
e) $f(s) = \frac{3e^{-4s}}{s(s-3)(s-5)}$

7. Find the Laplace transform of the function
$F(t) = t^2 + (t^2 - 4)\alpha(t-5)$.

8. Find the Laplace transform of the solution to the DE

$x''(t) - 25x(t) = F(t)$

where $x(0) = 0,\ x'(0) = 2,$ and

$$F(t) = \begin{cases} 6 \text{ for } 0 < t < 3 \\ t \text{ for } t \geq 3 \end{cases}$$

So just find $\boldsymbol{f(s)}$. Do not find $\boldsymbol{L}^{-1}(f(s))$.

Solutions

1. $(8xy + 5y^3)dx + (4x^2 + 15xy^2 + 2y)dy = 0$

$\dfrac{\partial M}{\partial y} = 8x + 15y^2$ and $\dfrac{\partial N}{\partial x} = 8x + 15y^2$

The DE is exact.

$F(x, y) = 4x^2y + 5y^3x + y^2$

and the solution is $\boldsymbol{4x^2y + 5y^3x + y^2 = C}$.

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 6y = 0$

$m^2 - 2m + 6 = 0 \;\Rightarrow\; m^2 - 2x + 1 = -6 + 1 \;\Rightarrow$

$(m-1)^2 = -5 \;\Rightarrow\; m = 1 \pm i\sqrt{5}$.

The solution to the DE is

$$\boldsymbol{y = e^x(C_1 sin\sqrt{5}\,x + C_2 cos\sqrt{5}x}.$$

3. $\dfrac{dy}{dx} - \dfrac{1}{x}y = xy^3$

$y^{-3}\dfrac{dy}{dx} - \dfrac{1}{x}y^{-2} = x$

$\boldsymbol{w = y^{-2}} \;\Rightarrow\; \dfrac{dw}{dx} = -2y^{-3}\dfrac{dy}{dx}$

Mult. by -2.

$-2y^{-3}\dfrac{dy}{dx} + \dfrac{2}{x}y^{-2} = -2x$

$\dfrac{dw}{dx} + \dfrac{2}{x}w = -2x$

$$u = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$y^{-2} x^2 = \int (-2x) x^2 dx = -\int 2x^3 dx = -\frac{1}{2} x^4 + C$$

The solution is $\boldsymbol{y^{-2} x^2 = -\frac{1}{2} x^4 + C}$

or $2x^2 = -x^4 y^2 + C y^2$

4. $$\frac{d^2 y}{dx^2} + 36y = sin\, 6x$$

$m^2 + 36 = 0 \quad \Rightarrow \quad m^2 = -36 \quad \Rightarrow \quad m = \pm 6i$.

$y_c = C_1 sin\, 6x + C_2 cos\, 6x$

$y_p = kx\, cos\, 6x$

$y'_p = kcos 6x - 6kxsin\, 6x$

$y''_p = -6kin\, 6x - 6ksin\, 6x - 36kxcos\, 6x =$

$-12ksin\, 6x - 36kxcos\, 6x$

$(D^2 + 36) y_p = -12ksin\, 6x - 36kxcos\, 6x + 36kxcos\, 6x =$

$= -12ksin\, 6x$ which must equal $sin\, 6x \quad \Rightarrow$

$-12k = 1 \quad \Rightarrow \quad k = -1/12$.

Therefore $x = C_1 sin\, 6x + C_2 cos\, 6x - \frac{1}{12} kxcos\, 6x$.

5. a) $F(t) = 5t^2 + cos\, 8t + sin\, 10t$

$$L(F(t)) = \frac{10}{s^3} + \frac{s}{s^2 + 64} + \frac{10}{s^2 + 100}$$

b) $F(t) = e^{-4t} t^2$

$L(F(t)) = f(s+4)$ where $f(s) = L^{-1}(t^2) = \frac{2}{s^3}$

Therefore $L(F(t)) = \frac{2}{(s+4)^3}$.

c) $F(t) = e^{9t} cos\, 7t$

$L(F(t)) = f(s-9)$ where $f(s) = L^{-1}(cos\, 7t) = \frac{s}{s^2 + 49}$

Therefore $L(F(t)) = \frac{s-9}{(s-9)^2 + 49}$.

6. a) $L^{-1}\left(\frac{1}{s^2}+\frac{4}{(s-2)^2}\right)=t+4e^{2t}L^{-1}\left(\frac{1}{s^2}\right)=t=\ \boldsymbol{t+4te^{2t}}.$

b) $L^{-1}\left(\frac{3}{(s+2)^2+25}\right)=e^{-2t}L^{-1}\left(\frac{3}{s^2+25}\right)=\boldsymbol{\frac{3}{5}e^{-2t}sin\,5t}.$

c) $L^{-1}\left(\frac{3e^{-4s}}{s^2+36}\right)=\alpha(t-4)G(t-4)$ where

$G(t)=L^{-1}\left(\frac{3}{s^2+36}\right)=\frac{3}{6}sin(6t).$

Ans: $\frac{1}{2}\alpha(t-4)sin(6[t-4])$ or

$$\boldsymbol{\frac{1}{2}\alpha(t-4)sin(6t-24)}$$

d) $f(s)=\frac{s+9}{s^2(s^2+4)}=\frac{s}{s^2(s^2+4)}+\frac{9}{s^2(s^2+4)}=$

$$\frac{1}{s(s^2+4)}+\frac{9}{s^2(s^2+4)}=$$

$$\frac{1/4}{s}-\frac{(1/4)s}{s^2+4}+\frac{9/4}{s^2}-\frac{9/4}{s^2+4}\ .$$

Therefore $L(f(s))=\frac{1}{4}-\frac{1}{4}cos\,2t+\frac{9}{4}t-\frac{9}{8}sin\,2t.$

e) $f(s)=\frac{3e^{-4s}}{s(s-3)(s-5)}$

$\frac{3}{s(s-3)(s-5)}=\frac{a}{s}+\frac{b}{s-3}+\frac{c}{s-5}$

$3=a(s-3)(s-5)+bs(s-5)+cs(s-3)$

$s=0\ \Rightarrow\ 3=15a\ \Rightarrow\ a=1/5.$

$s=3\ \Rightarrow\ 3=b(3)(-2)\ \Rightarrow\ b=-1/2.$

$s=5\ \Rightarrow\ 3=c(5)(2)\ \Rightarrow\ c=3/10.$

$L^{-1}(f(s))=\alpha(t-4)G(t-4)$ where

$$G(t) = L\left(\frac{1/5}{s} + \frac{-1/2}{s-3} + \frac{3/10}{s-5}\right) =$$
$$\frac{1}{5} - \frac{1}{2}e^{3t} + \frac{3}{10}e^{5t}\,.$$

Therefore $L^{-1}(f(s)) =$
$$\alpha(t-4)\left[\frac{1}{5} - \frac{1}{2}e^{3(t-4)} + \frac{3}{10}e^{5(t-4)}\right].$$

7. $F(t) = t^2 + (t^2-4)\alpha(t-5)$

$L(F(t)) = \dfrac{2}{s^2} + e^{-5s}L(G(t))$ where

$G(t-5) = t^2 - 4 \quad \Rightarrow \quad G(t) = (t+5)^2 - 4 = t^2 + 10t + 21\,.$

Therefore $L(F(t)) = \dfrac{2}{s^3} + e^{-5s}\left(\dfrac{2}{s^3} + \dfrac{10}{s^2} + \dfrac{21}{s}\right).$

8. $F(t) = 6 - 6\alpha(t-3) + t\alpha(t-3) = 6 + \alpha(t-3)(t-6)$

$L(F(t)) = \dfrac{6}{s} + e^{-3s}L(G(t))$ where

$G(t-3) = t - 6 \quad \Rightarrow \quad G(t) = (t+3) - 6 = t - 3$.

Therefore $L(F(t)) = \dfrac{6}{s} + e^{-3s}L(t-3) = \dfrac{6}{s} + e^{-3s}\left(\dfrac{1}{s^2} - \dfrac{3}{s}\right)$

$L(x'' - 25x) = L(F(t))$

$$s^2f(s) - s\cdot 0 - 2 - 25f(s) = \frac{6}{s} + e^{-3s}\left(\frac{1}{s^2} - \frac{3}{s}\right)$$

$f(s)[\,s^2 - 25] = 2 + \dfrac{6}{s} + e^{-3s}\left(\dfrac{1}{s^2} - \dfrac{3}{s}\right)$ or

$$\frac{2s+6}{s} + e^{-3s}\left(\frac{1-3s}{s^2}\right)$$

$$f(s) = \frac{2}{s^2-25} + \frac{6}{s(s^2-25)} + e^{-3s}\left(\frac{1}{s^2(s^2-25)} - \frac{3}{s(s^2-25)}\right)$$

or $\dfrac{2s+6}{s(s^2-25)} + e^{-3s}\left(\dfrac{1-3s}{s^2(s^2-25)}\right).$