Practice Test 3/Final Exam

1. Find the Laplace transform.
   a) $t^2 + 5t + 3$ b) $4\,sin\,7t + 2\,cos\,4t$
   c) $t^2 e^{5t}$ d) $e^{-3t} sin\,6t$

2. Find the inverse transform.
   a) $\frac{5}{s} + \frac{6}{s^2} + \frac{8}{s^3}$ b) $\frac{3}{(s-4)^2 + 25}$
   c) $\frac{5}{s^2 + 2s + 5}$ d) $\frac{2s}{(s-4)^2 + 25}$
   e) $\frac{2s}{s^2 + 10s + 34}$ f) $\frac{2s^2 + 1}{s(s+1)^2}$

3. a) Find the Laplace transform of $2\,t\alpha(t-6)$.
   b) Find the Laplace transform of $F(t)$ where

$$F(t) - \begin{cases} t \text{ for } 0 \le t < 4 \\ 5 \text{ for } t \ge 4 \end{cases}$$

   c) Find the inverse transform of $\frac{e^{-6s}}{s^2}$.
   d) Find the inverse transform of $\frac{e^{-3s}}{s^2 + 16}$.
   e) Find the Laplace transform of $F(t)$ where

$$F(t) = \begin{cases} 1 \text{ for } 0 \le t < 2 \\ t^2 \text{ for } t \ge 2 \end{cases}$$

4. Solve the DE by using Laplace transforms.
   a) $y''(t) - 5y'(t) + 4y(t) = 0$ where $y(0) = 5,\ y'(0) = 0$
   b) $y''(t) + y'(t) - 2y(t) = -4$ where $y(0) = 2,\ y'(0) = 3$

5.. A spring is such that it would be stretched 6 inches by a 12 pound weight. Let the weight be attached to the spring and pulled down 4 inches below the equilibrium point. If the weight is started with an upward velocity of 2 ft/sec, and assuming there is no damping force or impressed force present, answer the following questions:

a) What is the equation describing the motion?
b) What is the amplitude?
c) How long does it take for the weight to get back to the equilibrium point?
d) How long does it take for the weight to come back down again to its lowest point?

6. Solve the DE.

a) $\dfrac{dy}{dx} - \dfrac{1}{x}y = xy^3$

b) $\dfrac{d^2y}{dx^2} + 20\dfrac{dy}{dx} + 100y = 5$

c) $\dfrac{d^2y}{dx^2} + 100y = 2\,cos\,10x$

d) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 4e^{3x}$

7. Find the Laplace transform of the solution to the DE. Do not get the inverse transform.

a) $\dfrac{d^2x}{dt^2} + 16x = cos\,2t;\ x(0) = 2,\ x'(0) = 1$

b) $\dfrac{d^2x}{dt^2} + 9x = F(t)$

where $F(t) = \begin{cases} 0 \text{ for } 0 \le t < 6 \\ t - 2 \text{ for } t \ge 6 \end{cases}$

and $x(0) = 5,\ x'(0) = 0.$

Solutions

1. a) $L(t^2 + 5t + 3) = \frac{2}{s^3} + \frac{5}{s^2} + \frac{3}{s}$

   b) $L(4\,sin\,7t + 2\,cos\,4t) = \frac{28}{s^2 + 49} + \frac{2s}{s^2 + 16}$

   c) $L(t^2 e^{5t}) = f(s-5)$ where $f(s) = L(t^2) = \frac{2}{s^3}$.

   Therefore $L(t^2 e^{5t}) = \frac{2}{(s-5)^3}$.

   d) $L(e^{-3t} sin\,6t) = f(s+3)$ where $f(s) = L(sin\,6t) =$ $\frac{6}{s^2+36}$. Therefore $L\left(e^{-3t} sin\,6t\right) = \frac{6}{(s+3)^2 + 36}$.

2. a) $L^{-1}\left(\frac{5}{s} + \frac{6}{s^2} + \frac{8}{s^3}\right) = \mathbf{5 + 6t + 4t^2}$

   b) $L^{-1}\left(\frac{3}{(s-4)^2 + 25}\right) = 3e^{4t} L^{-1}\left(\frac{1}{s^2+25}\right) =$
   $\frac{3}{5} e^{4t} L^{-1}\left(\frac{5}{s^2+25}\right) = \boldsymbol{\frac{3}{5} e^{4t} sin\,5t}.$

   c) $\frac{5}{s^2 + 2s + 5} = \frac{5}{(s+1)^2 + 4}$
   $L^{-1}\left(\frac{5}{(s+1)^2+4}\right) = \frac{5}{2} L^{-1}\left(\frac{2}{(s+1)^2+4}\right) =$
   $\frac{5}{2} e^{-t} L^{-1}\left(\frac{2}{s^2+4}\right) = \boldsymbol{\frac{5}{2} e^{-t} sin\,2t}.$

   d) $\frac{2s}{(s-4)^2 + 25} = \frac{2(s-4) + 8}{(s-4)^2 + 25}$.
   $L^{-1}\left(\frac{2s}{(s-4)^2+25}\right) =$
   $L^{-1}\left(\frac{2(s-4)}{(s-4)^2+25}\right) + \frac{8}{5} L^{-1}\left(\frac{5}{(s-4)^2+25}\right) =$
   $2\,e^{4t} L^{-1}\left(\frac{s}{s^2+25}\right) + \frac{8}{5} e^{4t} L^{-1}\left(\frac{5}{s^2+25}\right) =$
   $\boldsymbol{2e^{4t} cos\,5t + \frac{8}{5} e^{4t} sin\,5t}.$

e) $$\frac{2s}{s^2+10s+34} = \frac{2s}{(s+5)^2+9} = \frac{2(s+5)-10}{(s+5)^2+9}$$

$$L^{-1}\left(\frac{2s}{s^2+10s+34}\right) =$$
$$2L^{-1}\left(\frac{s+5}{(s+5)^2+9}\right) - \frac{10}{3}L^{-1}\left(\frac{3}{(s+5)^2+9}\right) =$$
$$2e^{-5t}L^{-1}\left(\frac{s}{s^2+9}\right) - \frac{10}{3}e^{-5t}L^{-1}\left(\frac{3}{s^2+9}\right) =$$
$$\boldsymbol{2e^{-5t}\cos 3t - \frac{10}{3}e^{-5t}\sin 3t}.$$

f) $$\frac{2s^2+1}{s(s+1)^2} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$$

$2s^2+1 = a(s+1)^2 + bs(s+1) + cs$
If $s=0$ then $1=a$.
If $s=-1$ then $3=-c \Rightarrow c=-3$.
If $s=1$ then $3=4a+2b+c \Rightarrow 3=4+2b-3$
$\Rightarrow 2b=2 \Rightarrow b=1.$
The inverse transform of $\frac{1}{s} + \frac{1}{s+1} + \frac{-3}{(s+1)^2}$ is

$\boldsymbol{1+e^{-t}-3te^{-t}}$.

3. a) $L(2\ t\alpha(t-6)) = 2L(t\,\alpha(t-6)) = 2e^{-6s}L(F(t))$ where
$F(t-6)=t \Rightarrow F(t)=t+6$ and $L(F(t)) = \frac{1}{s^2} + \frac{6}{s}$.
Therefore $L(2\ t\alpha(t-6)) = \boldsymbol{2e^{-6s}\left(\frac{1}{s^2}+\frac{6}{s}\right)}$.

b) $F(t) = t - t\,\alpha(t-4) + 5\,\alpha(t-4) =$
$t+\alpha(t-4)(-t+5) = t-\alpha(t-4)(t-5).$

$L(F(t)) = L(t) - L(\alpha(t-4)(t-5)) = \frac{1}{s^2} - e^{-4s}L(G(t))$
where $G(t-4)=t-5 \Rightarrow G(t)=(t+4)-5=t-1.$
Therefore $L(F(t)) = \frac{1}{s^2} - e^{-4s}\left(\frac{1}{s^2}-\frac{1}{s}\right)$ or

$$\boldsymbol{\frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)}.$$

c) $L^{-1}\left(\dfrac{e^{-6s}}{s^2}\right) = \alpha(t-6)F(t-6)$ where

$F(t) = L^{-1}\left(\dfrac{1}{s^2}\right) = t$ and $F(t-6) = t-6$.

Therefore $L^{-1}\left(\dfrac{e^{-6s}}{s^2}\right) = \boldsymbol{\alpha(t-6)(t-6)}$.

d) $L^{-1}\left(\dfrac{e^{-3s}}{s^2+16}\right) = \alpha(t-3)F(t-3)$ where

$F(t) = L^{-1}\left(\dfrac{1}{s^2+16}\right) = \dfrac{1}{4} sin\, 4t$.

Therefore $L^{-1}\left(\dfrac{e^{-3s}}{s^2+16}\right) = \dfrac{1}{4}\alpha(t-3) sin(4[t-3]) =$

$$\boldsymbol{\frac{1}{4}\alpha(t-3)\, sin(4t-12)}.$$

e) $F(t) = \begin{cases} 1 \text{ for } 0 \le t < 2 \\ t^2 \text{ for } t \ge 2 \end{cases}$

$$F(t) = 1 - 1\,\alpha(t-2) + t^2\,\alpha(t-2) = 1 + \alpha(t-2)(t^2-1).$$

$$L(F(t)) = L(1) + L(\alpha(t-2)(t^2-1)) =$$

$$\frac{1}{s} + e^{-2s} L\big(([t+2]^2 - 1)\big) =$$

$$\frac{1}{s} + e^{-2s} L\big(t^2 + 4t + 4 - 1\big) =$$

$$\frac{1}{s} + e^{-2s} L\big(t^2 + 4t + 3\big) =$$

$$\boldsymbol{\frac{1}{s} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right)}.$$

4. a) $y'' - 5y' + 4y = 0$ where $y(0) = 5,\ y'(0) = 0$

$$L(y'') - 5L(y') + 4L(y) = L(0)$$
$$s^2 f(s) - s \cdot y(0) - y'(0) - 5(s \cdot f(s) - y(0)) + 4f(s) = 0$$
$$s^2 f(s) - 5s - 5sf(s) + 25 + 4f(s) = 0$$
$$f(s)(s^2 - 5s + 4) = 5s - 25$$
$$f(s) = \frac{5s - 25}{(s-4)(s-1)} = \frac{a}{s-4} + \frac{b}{s-1}$$
$$5s - 25 = a(s-1) + b(s-4)$$
$$s = 4 \ \Rightarrow\ -5 = 3a \ \Rightarrow\ a = -5/3$$
$$s = 1 \ \Rightarrow\ -20 = -3b \ \Rightarrow\ b = 20/3$$

$$f(s) = \frac{-5/3}{s-4} + \frac{20/3}{s-1}$$ ← *this is the inverse transform of the solution*

$$x(t) = L^{-1}\left(\frac{-5/3}{s-4} + \frac{20/3}{s-1}\right) = -\mathbf{\frac{5}{3}}\boldsymbol{e^{4t}} \boldsymbol{+} \mathbf{\frac{20}{3}}\boldsymbol{e^{t}}$$

b) $y'' + y' - 2y = -4$ where $y(0) = 2,\ y'(0) = 3$

$$L(y'') + L(y') - 2L(y) = -L^{-1}(4)$$

$$s^2 f(s) - s \cdot y(0) - y'(0) + sf(s) - y(0) - 2f(s) = -4/s$$

$$s^2 f(s) - 2s - 3 + sf(s) - 2 - 2f(s) = -4/s$$
$$\left(s^2 + s - 2\right) f(s) = 2s + 5 - \frac{4}{s} \text{ or } \frac{2s^2 + 5s - 4}{s}$$
Solve for $f(s)$ by dividing by $s^2 + s - 2$ or $(s+2)(s-1)$.
$$f(s) = \frac{2s^2 + 5s - 4}{s(s+2)(s-1)} = \frac{a}{s} + \frac{b}{s+2} + \frac{c}{s-1}$$
$$2s^2 + 5s - 4 = a(s+2)(s-1) + bs(s-1) + cs(s+2)$$
$$s = 0 \ \Rightarrow\ -4 = -2a \ \Rightarrow\ a = 2$$
$$s = -2 \ \Rightarrow\ 8 - 10 - 4 = -6 = b(6) \ \Rightarrow\ b = -1$$
$$s = 1 \ \Rightarrow\ 3 = c(3) \ \Rightarrow\ c = 1$$
$$f(s) = \frac{2s^2 + 5s - 4}{s(s+2)(s-1)} = \frac{2}{s} - \frac{1}{s+2} + \frac{1}{s-1}$$
$y = L^{-1}(f(s))$ and $\boldsymbol{y = 2 - e^{-2t} + e^{t}}$.

5. $12 = \frac{1}{2}k \quad \Rightarrow \quad k = 24$

$\frac{12}{32}x'' + 24x = 0$

$x'' + 64x = 0$ where $x(0) = 1/3$ and $x'(0) = -2$

$x = a\,sin\,8t + b\,cos\,8t$ and $x' = 8a\,cos\,8t - 8b\,sin\,8t$

$x(0) = 1/3 \quad \Rightarrow \quad b = 1/3$

$x'(0) = -2 \quad \Rightarrow \quad -2 = 8a \quad \Rightarrow \quad a = -1/4$

Therefore the equation of motion is

$$\boldsymbol{x = -\tfrac{1}{4}\,sin\,8t + \tfrac{1}{3}\,cos\,8t.}$$

The amplitude is A = $\sqrt{1/16 + 1/9} = \sqrt{25/144} =$ **5/12 of a foot.**

$x = \frac{5}{12}sin(8t + \phi)$ where ϕ is located in the quadrant containing the point $\left(-\frac{1}{4}, \frac{1}{3}\right)$, the 2nd quadrant, and $\phi = \pi - tan^{-1}\left(\frac{1/3}{1/4}\right) = 2.214$.

So $\boldsymbol{x = \frac{5}{12}sin(8t + 2.214).}$

When the weight gets back to the equilibrium point,

$\frac{5}{12}sin(8t + 2.214) = 0 \quad \Rightarrow \quad 8t + 2.214 = 0$ or π or 2π, $\cdots$.

Because we want to find the 1st positive value of t where this occurs, we want $8t + 2.214 = \pi \quad \Rightarrow$

$$t = \frac{\pi - 2.214}{8} = \mathbf{0.116}\text{ seconds.}$$

When the weight comes back down again, x is at its maximum which implies that $8t + 2.214 = \pi/2$, $5\pi/2$, $9\pi/2$, ... We want the 1st positive value of t where this occurs. Therefore

$8t + 2.214 = 5\pi/2 \quad \Rightarrow \quad t = \frac{5\pi/2 - 2.214}{8} = \mathbf{0.705}$ **seconds**.

6. a) $\dfrac{dy}{dx} - \dfrac{1}{x}y = xy^3$ is Bournoulli's equation.

$$y^{-3}\frac{dy}{dx} - \frac{1}{x}y^{-2} = x$$

$$\boldsymbol{w = y^{-2}} \quad \Rightarrow \quad \frac{dw}{dx} = -2y^{-3}\frac{dy}{dx}$$

Mult. by -2.

$$-2y^{-3}\frac{dy}{dx} + \frac{2}{x}y^{-2} = -2x$$

$$\frac{dw}{dx} + \frac{2}{x}w = -2x$$

$$u = e^{\int \frac{2}{x}dx} = e^{2ln\,x} = x^2$$

$$y^{-2}x^2 = \int(-2x)x^2dx = -\int 2x^3\,dx = -\frac{1}{2}x^4 + C$$

The solution is $\boldsymbol{y^{-2}x^2 = -\frac{1}{2}x^4 + C}$

or $\boldsymbol{2x^2 = -x^4y^2 + Cy^2}$.

b) $\dfrac{d^2y}{dx^2} + 20\dfrac{dy}{dx} + 100y = 5$

$m^2 + 20m + 100 = 0 \quad \Rightarrow \quad (m+10)^2 = 0 \quad \Rightarrow$

-10 is a root of multiplicity 2 $\quad \Rightarrow$

$y_c = C_1e^{-10x} + C_2xe^{-10x}$.

$y_p = k \quad \Rightarrow \quad (D^2 + 20D + 100)k = 100k =$ (?) 5

$\Rightarrow \quad k = 1/20$.

The general solution is

$$y = C_1e^{-10x} + C_2xe^{-10x} + \frac{1}{20}.$$

c) $\dfrac{d^2y}{dx^2} + 100y = 2\,cos\,10x$

$m^2 + 100 = 0 \;\Rightarrow\; m = \pm 10\,i \;\Rightarrow$
$y_c = C_1 cos\,10x + C_2 sin\,10x.$

$y_p = kx\,sin\,10x$
$y_p' = k\,sin\,10x + 10kx\,cos\,10x$
$y_p'' = 10\,k\,cos\,10x + 10k\,cos\,10x - 100kx\,sin\,10x$
$= 20k\,cos\,10x - 100kx\,sin\,10x\,.$

$(D^2 + 100)y_p =$
$20k\,cos\,10x - 100kx\,sin\,10x + 100ks\,sin\,10x =$
$20k\,cos\,10x = (?)\,2 \;\Rightarrow\; k = 2/20 = 1/10.$

The general solution is

$$y = C_1 cos\,10x + C_2 sin\,10x + \frac{1}{10}x\,sin\,10x\,.$$

d) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 4e^{3x}$

$m^2 - 4m + 3 = 0 \;\Rightarrow\; (m-3)(m-1) = 0 \;\Rightarrow$
$m = 3,\ 1 \;\Rightarrow\; y_c = C_1 e^{3x} + C_2 e^x.$
$y_p = kxe^{3x} \quad \leftarrow \times 3$
$y_{p'} = ke^{3x} + 3kxe^{3x} \leftarrow \times\ -4$
$y_p'' = 3ke^{3x} + 3ke^{3x} + 9kxe^{3x} = 6ke^{3x} + 9kxe^{3x} \leftarrow \times 1$
$(D^2 - 4D + 3)y_p = 6ke^{3x} + 9kxe^{3x}$
$- 4ke^{3x} - 12kxe^{3x}$
$+ 3ke3x$
$= 2ke^{3x} = (?)\,4e^{3x} \;\Rightarrow\; k = 2.$

The general solution is

$$y = C_1 e^{3x} + C_2 e^x + 2x\,e^{3x}.$$

7. a) $\dfrac{d^2x}{dt^2} + 16x = cos\,2t;\ x(0) = 2,\ x'(0) = 1$

$L(x'') + 16L(x) = L(cos\,2t)$
$s^2 f(s) - s \times 2 - 1 + 16f(s) = \dfrac{s}{s^2+4}$
$s^2 f(s) - 2s - 1 + 16f(s) = \dfrac{s}{s^2+4}$

$$f(s)(s^2+16) = 1 + 2s + \frac{s}{s^2+4}$$

$$f(s) = \frac{1}{s^2+16} + \frac{2s}{s^2+16} + \frac{s}{(s^2+4)(s^2+16)}$$

Note that $\dfrac{s}{(s^2+4)(s^2+16)} = \dfrac{s}{12(s^2+4)} - \dfrac{s}{12(s^2+16)}$

and $x(t) = \dfrac{23}{12} cos\, 4t + \dfrac{1}{4} sin\, 4t + \dfrac{1}{12} cos\, 2t$.

b) $\dfrac{d^2x}{dt^2} + 9x = F(t)$

where $F(t) = \begin{cases} 0 \text{ for } 0 \le t < 6 \\ t-2 \text{ for } t \ge 6 \end{cases}$

and $x(0) = 5,\ x'(0) = 0.$

$F(t) = (t-2)\alpha(t-6)$ and

$$L((t-2)\alpha(t-6)) = L((t-6+4)\alpha(t-6)) = e^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right).$$

$$L(x'') + 9L(x) = L(F(t))$$

$$s^2 f(s) - 5s + 9f(s) = e^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

$$f(s)\big(s^2+9\big) = 5s + e^{-6s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

$$f(s) = \frac{5s}{s^2+9} + e^{-6s}\left(\frac{1}{s^2(s^2+9)} + \frac{4}{s(s^2+9)}\right)$$