Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 7.2 (p. 385)- Definition of Laplace Transform

1. $$L(t) = \int_0^{\infty} e^{-st} t\, dt = \left. \frac{-\, te^{-st}}{s} \right]_0^{\infty} + \int_0^{\infty} \frac{e^{-st}}{s} dt =$$

where $u = t,\ dv = e^{-st}dt$

$$du = dt,\ v = \frac{-\, e^{-xt}}{s}$$

$$0 + \left. \frac{e^{-st}}{-\, s^2} \right]_0^{\infty} = \frac{1}{s^2}\,.$$

3. $$L(e^{6t}) = \int_0^{\infty} e^{-st} e^{6t} dt = \int_0^{\infty} e^{(6-s)t} dt = \left. \frac{e^{(6-s)t}}{6-s} \right]_0^{\infty} =$$

assuming $s > 6$

$$0 - \frac{e^0}{6-s} = \frac{1}{s-6}\,.$$

4. $$L(te^{3t}) = \int_0^{\infty} e^{-st} te^{3t} dt = \int_0^{\infty} te^{(3-s)t} dt =$$

where $u = t,\ dv = e^{(3-s)t}dt$

$$du = dt,\ v = \frac{e^{(3-s)t}}{3-s}$$

$$t\left. \frac{e^{(3-s)t}}{3-s} \right]_0^{\infty} - \frac{1}{3-s} \int_0^{\infty} e^{(3-s)t} dt =$$

assuming $s > 3$

$$0 - \frac{1}{3-s} \left. \frac{e^{(3-s)t}}{3-s} \right]_0^{\infty} = \left. \frac{e^{(3-s)t}}{(3-s)^2} \right]_{\infty}^{0} = \frac{1}{(s-3)^2}\,.$$

9. $f(t) = \begin{cases} 0 \text{ if } 0 < t < 2 \\ t \text{ if } t > 2 \end{cases}$

$$L(f(t)) = \int_2^\infty e^{-st} t dt = \frac{-te^{-st}}{s}\Big]_2^\infty + \frac{1}{s}\int_2^\infty e^{-st} dt$$

where $u = t,\ dv = e^{-st}dt$

$$du = dt,\ v = \frac{e^{-st}}{-s}$$

$$= \frac{te^{-st}}{s}\Big]_\infty^2 - \frac{e^{-st}}{s^2}\Big]_2^\infty = \frac{2e^{-2s}}{s} - 0 + \frac{e^{-st}}{s^2}\Big]_\infty^2 =$$

$$\frac{2e^{-2s}}{s} + \frac{e^{-2s}}{s^2} = e^{-2s}\left(\frac{2}{s} + \frac{1}{s^2}\right) = e^{-2s}\left(\frac{2s+1}{s^2}\right).$$

12. $f(t) = \begin{cases} e^{2t} \text{ if } 0 < t < 3 \\ 1 \text{ if } t > 3 \end{cases}$

$$L(f(t)) = \int_0^3 e^{-st}e^{2t}dt + \int_3^\infty e^{-st}dt =$$

$$\int_0^3 e^{(2-s)t}dt + \int_3^\infty e^{-st}dt = \frac{e^{(2-s)t}}{2-s}\Big]_0^3 + \frac{e^{-st}}{s}\Big]_\infty^3 =$$

$$\frac{e^{(2-s)3}}{2-s} - \frac{1}{2-s} + \frac{e^{-3s}}{s} =$$

$$\frac{1}{s-2} + \frac{e^{-3s}}{s} - \frac{e^6 e^{-3s}}{s-2}.$$

21. $f(t) = \begin{cases} 1 & \text{for } 0 \le t \le 1 \\ (t-2)^2 & \text{for } 1 < t \le 10 \end{cases}$

This function is continuous $\Rightarrow$ it is peicewise continuous.

$\lim_{t \to 1^+} (t-2)^2 = (-1)^2 = 1$ and

$$\lim_{t \to 1^-} 1 = 1.$$

23. $f(t) = \begin{cases} 1 & \text{for } 0 \le t < 1 \\ t-1 & \text{for } 1 < t < 3 \\ t^2 - 4 & \text{for } 3 < t \le 10 \end{cases}$

This function is not continuous but is is peicewise continuous.

$\lim_{t \to 3^-} (t-1) = 3 - 1 = 2$ but

$\lim_{t \to 3^+} (t^2 - 4) = 9 - 4 = 5.$

Remember, a function is piecewise continuous on a finite interval [a,b] if it is continuous for all t except for a finite number of jump discontinuities.

At $x = 3$, there is a jump discontinuity.

27. $f(t) = \begin{cases} 1/t & \text{for } 0 < t < 1 \\ 1 & \text{for } 1 \le t \le 2 \\ 1 - t & \text{for } 2 < t \le 10 \end{cases}$

Neither continuous nor piecewise continuous

Note that at $t = 0$ there is a limit of infinity!