Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 4.9 (p. 238)- Free Mechanical Vibrations

1. $my'' + by' + cy = 0$
$3y'' + 48y = 0 \Rightarrow y'' + 16y = 0 \Rightarrow$
$y = acos\,4t + bsin4t$
$y(0) = -\frac{1}{2} \Rightarrow a = -\frac{1}{2}.$
$y' = -4asin4t + 4bcos4t$
$y'(0) = 2 = 4b \Rightarrow b = \frac{1}{2}$
$y = -\frac{1}{2}cos\,4t + \frac{1}{2}sin\,4t = \sqrt{\frac{1}{4} + \frac{1}{4}}\,sin\,(4t + \phi)$
where ϕ is quadrant containing the point $\left(\frac{1}{2}, -\frac{1}{2}\right)$ which is 4.
Therefore $\phi = -tan^{-1}(1) = -\pi/4$
Ans: $\boldsymbol{y = \frac{\sqrt{2}}{2}\,sin\left(4t - \frac{\pi}{4}\right)}$

3. There are 4 parts.
Part 1= $b = 0$
$y'' + 16y = 0$ where $y(0) = 1$ and $y'(0) = 0$
$y = a\,cos\,4t + b\,sin\,4t$
$y(0) = 1 \Rightarrow a = 1$
$y' = -4asin\,4t + 5bcos\,4t$ and $y'(0) = 0 \Rightarrow b = 0$
$y = cos\,4t$

Part 2- $b = 6$
$y'' + 6y' + 16y = 0 \Rightarrow m = -3 \pm i\sqrt{7}$
$y = e^{-3t}\left(a\,cos\,\sqrt{7}\,t + b\,sin\,\sqrt{7}\,t\right)$ where $y(0) = 1$ and $y'(0) = 0$
$y(0) = 1 = a$
$y' = -3e^{-3t}\left(a\,cos\,\sqrt{7}\,t + b\,sin\,\sqrt{7}\,t\right) +$
$e^{-3t}\left(-\sqrt{7}\,a\,sin\,\sqrt{7}\,t + \sqrt{7}\,b\,cos\,\sqrt{7}\,t\right)$
$y'(0) = -3\,a + \sqrt{7}\,b = -3 + \sqrt{7}\,b \Rightarrow b = \frac{3}{\sqrt{7}}$
$y = e^{-3t}\left(cos\,\sqrt{7}\,t + \frac{3}{\sqrt{7}}\,sin\,\sqrt{7}\,t\right) =$

$$\sqrt{1+\frac{9}{7}}\, sin\left(\sqrt{7}\,t+\phi\right) \text{ where } \phi = tan^{-1}\left(\frac{\sqrt{7}}{3}\right)$$

Part 3- $b = 8$

$y'' + 8y' + 16y = 0 \quad \Rightarrow \quad m =$

$(m+4)^2 = 0 \quad \Rightarrow \quad m = -4$ is a root of mult. 2

$y = a\,e^{-4t} + bte^{-4t}$

$y(0) = 1 \quad \Rightarrow \quad a = 1$

$y' = -4ae^{-4t} + be^{-4t} - 4bte^{-4t}$

$y'(0) = 0 = -4a + b \quad \Rightarrow b = 4$

$y = e^{-4t} + 4te^{-4t} = (1+4t)e^{-4t}$

Part 4- $b = 10$

$y'' + 10y' + 16y = 0$

$m^2 + 10m + 16 = 0 \quad \Rightarrow \quad m = -8$ or -2

$y = ae^{-8t} + be^{-2t}$

$y(0) = 1 \; = a + b$

$y' = -8ae^{-8t} - 2be^{-2t}$

$y'(0) = 0 = -8a - 2b$

$2 = \; 2a + 2b$

$\Rightarrow 2 = -6a \quad \Rightarrow \quad a = -1/3$

$-\frac{1}{3} + b = 1 \quad \Rightarrow \quad b = 1 + \frac{1}{3} = \frac{4}{3}$

$y = -\frac{1}{3}e^{-8t} + \frac{4}{3}e^{-2t}$

7. $\frac{1}{8}y'' + 2y' + 16y = 0$ where $y(0) = -\frac{3}{4}$ and $y'(0) = -2$

$y'' + 16y' + 128 = 0 \quad \Rightarrow \quad m^2 + 16m = -128 \quad \Rightarrow$

$(m+8)^2 = -128 + 64 = -64 \quad \Rightarrow \quad m = -8 \pm 8i$

$y = e^{-8t}(a\,cos\,8t \, + \, b\,sin\,8t)$

$y(0) = -\frac{3}{4} \quad \Rightarrow \quad a = -\frac{3}{4}$

$y' = -8e^{-8t}(a\,cos\,8t \, + \, b\,sin\,8t) + e^{-8t}(-8asin8t + 8bcos8t)$

$y'(0) = -2 \; = -8a + 8b = -8\left(-\frac{3}{4}\right) + 8b$

$-2 = 6 + 8b \quad \Rightarrow \quad b = -1$

$$y = e^{-8t}\left(-\frac{3}{4} cos\, 8t - sin\, 8t\right) = \sqrt{\frac{9}{16}+1}\; e^{-8t}(sin\, 8t\; + \phi)$$

where ϕ is in quadrant 3 and $\phi = \pi + tan^{-1}(3/4)$

Damping factor is $\sqrt{25/16}\; e^{-8t} = 5/4e^{-8t}$.

Quasiperiod is π/4 and quasefreq. is 4/π.