Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 4.5 (p. 201)- The Superposition Principle

21. Find a general solution to the DE
$y''(\theta) + 2y'(\theta) + 2y(\theta) = e^{-\theta} cos\,\theta$
Note: this is a hard problem.

We will need the θ factor if $-1 \pm i$ is a solution to the auxiliary equation.

$m^2 + 2m + 2 = 0 \;\Rightarrow\; m^2 + 2x = -2$
$\Rightarrow\; (m+1)^2 = -1 \;\Rightarrow\; m = -1 \pm i$ and
$y_c = e^{-\theta}(C_1 cos\,\theta + C_2 sin\,\theta)$.

We need the θ factor. Note that terms with factors of θ will cancel out when we compute $y_p'' + 2y_p' + 2y_p$.
$y_p = \theta e^{-\theta}(a\,cos\,\theta + b\,sin\,\theta)$
$y_p' = e^{-\theta}(a\,cos\,\theta + b\,sin\,\theta) + \theta(e^{-\theta}[a\,cos\,\theta + b\,sin\,\theta])'$

$y_p'' = -e^{-\theta}(a\,cos\,\theta + b\,sin\,\theta) + e^{-\theta}(-a\,sin\,\theta + b\,cos\,\theta) +$
$(e^{-\theta}[a\,cos\,\theta + b\,sin\,\theta])' +$ terms with factor of θ
and we can ignore terms with factor of θ.

$y_p'' = -e^{-\theta}(a\,cos\,\theta + b\,sin\,\theta) + e^{-\theta}(-a\,sin\,\theta + b\,cos\,\theta)$
$-e^{-\theta}(a\,cos\,\theta + b\,sin\,\theta) + e^{-\theta}(-a\,sin\,\theta + b\,cos\,\theta) + \ldots$

$y_p'' = (-2a + 2b)e^{-\theta} cos\,\theta + (-2a - 2b)e^{-\theta} sin\,\theta +$ terms with θ factor will cancel out!

$$(D^2 + 2D + 2)y_p = \begin{pmatrix} -2a + 2b \\ 2a \\ 0 \end{pmatrix} e^{-\theta} cos\,\theta + \begin{pmatrix} -2a - 2b \\ 2b \\ 0 \end{pmatrix} e^{-\theta} sin\,\theta = \boldsymbol{(2b)e^{-\theta} cos\,\theta + (-2a)e^{-\theta} sin\,\theta}$$

which must equal the right side of DE $\boldsymbol{e^{-\theta} cos\,\theta} \;\Rightarrow$

$\mathbf{2b = 1} \;\Rightarrow\; b = 1/2$ and $\mathbf{-2a = 0} \;\Rightarrow\; a = 0$
$\Rightarrow\; y_p = \frac{1}{2}\theta\, e^{-\theta} sin\,\theta$.

Ans: $\boldsymbol{y = e^{-\theta}(C_1 cos\,\theta + C_2 sin\,\theta) + \frac{1}{2}\theta\, e^{-\theta} sin\,\theta}$

23. Find the specific solution to the DE
$y' - y = 1,\; y(0) = 0$

$m - 1 = 0 \;\; \Rightarrow \;\; y_c = ae^x.$
$y_p = k \;\; \Rightarrow \;\; (D-1)k = 1 \;\; \Rightarrow \;\; k = -1\,.$
$y = ae^x - 1 \;\; \Rightarrow \;\; 0 = a - 1 \;\; \Rightarrow \;\; a = 1\,.$
Ans: $y = e^x - 1$

25. Find the specific solution to $z'' + z = 2e^{-x}$ where $z(0) = 0$ and $z'(0) = 0.$

$z_c = a\,cos\,x + b\,sin\,x$
$z_p = ke^{-x}$ and $(D^2+1)ke^{-x} = ke^{-x} + ke^{-x} =$
$2ke^{-x} = 2e^{-x} \;\; \Rightarrow \;\; k = 1\,.$
$z = a\,cos\,x + b\,sin\,x + e^{-x}$
$z' = -a\,sin\,x + b\,cos\,x - e^{-x}$

$z(0) = a + 1 = 0 \;\; \Rightarrow \;\; a = -1\,.$
$z'(0) = b - 1 = 0 \;\; \Rightarrow \;\; b = 1.$
Ans: $y = sin\,x - cos\,x + e^{-x}$

31. Determine the form for the particular solution to the DE
$y'' + y = sin\,x + x\,cos\,x + 10^x$

$m^2 + 1 = 0 \;\; \Rightarrow \;\; m = \pm\, i\,.$
Since y_c involves $cos\,x$ and $sin\,x,\; y_p$ now we must include
$x sin\,x$ and $x\,cos\,x$ just to yield the $sin\,x$ on the right.
Furthermore, to yield the $x cos\,x$ on the right we need
$x^2 sin\,x$ and $x^2 cos\,x.$
Note that $10^x = e^{xln\,10} \;\; \Rightarrow \;\; y_p$ must have a e^{xln10} term.
Ans: $y_p = a\,x\,sin\,x + b\,x\,cos\,x + cx^2 sin\,x + dx^2 cos\,x + ke^{x\,ln10}$