Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 4.4 (p. 195)- Nonhomogeneous Eq's: The Method of Undetermined Coefficients

For questions 1, 3, and 5, does the method of undetermined coeff's apply to the DE?

1. $y'' + 2y' - y = t^{-1}e^t \leftarrow$ the method does not apply because t is raised to a negative power. NO

   The method applies if the right side of the DE involves functions like polynomials, $sn\,a\,x$, $cos\,ax$, e^{ax}, and products of these functions like $e^{2x}sin\,3x$ or x^2e^{4x}.

3. $x'' + 5x' - 3x = 3^t \leftarrow$ the method does apply because

   $$3^t = \left(e^{ln\,3}\right)^t = e^{(ln\,3)t}.$$

   and the right side has the form e^{kx}. YES

5. $y''(\theta) + 3y'(\theta) - y(\theta) = sec\,\theta \leftarrow$ the method does not apply because we would need a sin θ or cos θ on the right side.

   $sec\,\theta = \dfrac{1}{cos\,\theta}$ is no good.

   NO

17. Find a particular solution to $y'' - 2y' + y = 8e^t$.

    We must 1st see what y_c is equal to.

    $m^2 - 2m + 1 = 0 \;\Rightarrow\; (m-1)^2 = 0 \;\Rightarrow\; m = 1$ is a root of mult. 2 .

    $y_c = C_1e^t + C_2te^t$

    Because 1 is a root of mult. 2, we need a factor of t^2.

    Therefore
    $$y_p = k\,t^2e^t \quad \times 1$$
    $$y_p' = 2kte^t + kt^2e^t \quad \times -2$$
    $$y_p'' = 2ke^t + 2kte^t + 2kte^t + kt^2e^t =$$

$$2ke^t + 4kte^t + kt^2e^t \quad \times 1$$

$$(D^2 - 2D + 1)y_p = (2k)e^t + (4k - 4k)te^t + (k - 2k + k)t^2e^t$$
$$(2k)e^t + 0 + 0 = (?)\, 8e^t \quad \Rightarrow \quad k = 4\,.$$

Ans: $\boldsymbol{y_p = 4t^2e^t}$

21. Find a particular solution to $x'' - 4x' + 4x = te^{2t}$.

$m^2 - 4m + 4 = 0 \quad \Rightarrow \quad (m-2)^2 = 0 \quad \Rightarrow$ 2 is a root of mult. 2.
Normally $y_p = (a + bt)e^{2t}$ but now we need the t^2 factor and in place of $at + b$ we need $t^2(a + bt)$.

$$y_p = at^2e^{2t} + bt^3e^{2t} \quad \times 4$$

$$y_p' = 2ate^{2t} + 2at^2e^{2t} + 3bt^2e^{2t} + 2bt^3e^{2t} =$$
$$2ate^{2t} + (2a + 3b)t^2e^{2t} + 2bt^3e^{2t} \quad \times -4$$

$$y_p'' = 2ae^{2t} + 4ate^{2t}$$
$$+ (4a + 6b)te^{2t} + (4a + 6b)t^2e^{2t} +$$
$$6bt^2e^{2t} \qquad + 4bt^3e^{2t}$$

$$= 2ae^{2t} + (8a + 6b)te^{2t} + (4a + 12b)t^2e^{2t} + 4bt^3e^{2t} \quad \times 1$$

$$(D^2 - 4D + 4)y_p = (2a)e^{2t} + (8a + 6b - 8a)te^{2t} +$$

$$(4a + 12b - 8a - 12b + 4a)t^2e^{2t} +$$

$$(4b - 8b - 4b)t^3e^{2t} =$$

$2ae^{2t} + 6bte^{2t} + 0 + 0 = (?)\, te^{2t} \quad \Rightarrow \quad 2a = 0 \quad \Rightarrow \quad a = 0$
and $6b = 1 \quad \Rightarrow \quad b = 1/6$.

Ans: $\boldsymbol{y_p = \frac{1}{6}t^3e^{2t}}$

27. Determine the form of a particular solution to $y'' + 9y = 4t^3 sin\, 3t$.
Do not evaluate the coeff's.

$m^2 + 9 = 0 \;\Rightarrow\; m = \pm 3i$.
If $\pm 3i$ were not a solution, we could do with
$(at^3 + bt^2 + ct + d) sin\, 3t + (et^3 + ft^2 + gt + h) cos\, 3t,$
but now we must multiply by t.

Ans:
$y_p = (at^4 + bt^3 + ct^2 + dt) sin\, 3t + (et^4 + ft^3 + gt^2 + ht) cos\, 3t$

33. Find a particular solution to the DE
$y''' - y'' + y = sin\, t$

We must 1st check to see if i is a solution to the auxiliary equation
$m^3 - m^2 + m = 0$.
$i^3 - i^2 + i = (-i) + 1 + i = 1 \neq 0$

Therefore
$y_p = a\, sin\, t + b\, cos\, t \quad \times 1$
$y_p' = a\, cos\, t - b\, sin\, t \quad \times 0$
$y_p'' = -a\, sin\, t - b\, cos\, t \quad \times -1$
$y_p''' = -a\, cos\, t + b\, sin\, t \quad \times 1$

$(D^3 - D^2 + 1) y_p = (-a + b + b) cos\, t + (b + a + a) sin\, t$
$= (-a + 2b) cos\, t + (2a + b) sin\, t =$ (?) $sin\, t$
$\Rightarrow \quad -a + 2b = 0$ or $\quad -2a + 4b = 0$
$2a + b = 1$

$5b = 1 \;\Rightarrow\; b = 1/5$

$-a + 2/5 = 0 \;\Rightarrow\; a = 2/5$

Ans: $\boldsymbol{y_p = \frac{2}{5}\, sin\, t + \frac{1}{5}}$ **cos t**