Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 4.3 (p. 186)- Auxiliary Equations with Complex Roots

5. Find a general solution to the DE
$w'' + 4w' + 6w = 0$.

$m^2 + 4m + 6 = 0 \;\Rightarrow\; m^2 + 4m + 4 + 2 = 0 \;\Rightarrow$
$(m+2)^2 = -2 \;\Rightarrow\; m + 2 = \pm\sqrt{-2} \;\Rightarrow\; m = -2 \pm \sqrt{2}\, i.$
The general solution is $y = e^{-2t}\left(C_1 cos\,\sqrt{2}\,t + C_2 sin\,\sqrt{2}\,t\right)$.

17. Find a general solution to the equation
$y'' - y' + 7y = 0$.

$$m^2 - m + 7m = 0 \;\Rightarrow\; m = \frac{1 \pm \sqrt{1 - 4(1)(7)}}{2} = \frac{1 \pm \sqrt{-27}}{2} = \frac{1 \pm \sqrt{27}\, i}{2} = \frac{1 \pm 3\sqrt{3}\, i}{2} = \frac{1}{2} \pm \frac{3\sqrt{3}}{2} i.$$

The general solution is $y = e^{t/2}\left(C_1 cos\,\frac{\sqrt{3}}{2}t + C_2 sin\,\frac{\sqrt{3}}{2}t\right)$.

23. Solve the initial value problem
$w'' - 4w' + 2w = 0;\; w(0) = 0,\; w'(0) = 1.$

$m^2 - 4m + 2 = 0 \;\Rightarrow\; m^2 - 4m + 4 = -2 + 4 \;\Rightarrow$
$(m-2)^2 = 2 \;\Rightarrow\; m - 2 = \pm\sqrt{2} \;\Rightarrow\; m = 2 \pm \sqrt{2}$.

The general solution is $y = ae^{\left(2+\sqrt{2}\right)t} + be^{\left(2-\sqrt{2}\right)t}$.

$$y' = \left(2 + \sqrt{2}\right)ae^{\left(2+\sqrt{2}\right)t} + \left(2 - \sqrt{2}\right)be^{\left(2-\sqrt{2}\right)t}$$

$$y(0) = a + b = 0 \;\Rightarrow\; b = -a.$$

$$y'(0) = \left(2+\sqrt{2}\right)a + \left(2-\sqrt{2}\right)b = 1$$
$$\left(2+\sqrt{2}\right)a + \left(2-\sqrt{2}\right)(-a) = 1 \quad \Rightarrow$$
$$\left(2+\sqrt{2}-2+\sqrt{2}\right)a = 1 \quad \Rightarrow$$
$$2\sqrt{2}\,a = 1 \quad \Rightarrow \quad a = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}.$$

Since $b = -a,\ b = -\dfrac{\sqrt{2}}{4}$.

The specific solution is $y = \dfrac{\sqrt{2}}{4}e^{\left(2+\sqrt{2}\right)t} - \dfrac{\sqrt{2}}{4}e^{\left(2-\sqrt{2}\right)t}$.

29 Find a general solution to the DE

a) $y''' - y'' + y' + 3y = 0$.

$m^3 - m^2 + m + 3 = 0$

$m = -1$ is a solution.

Using synthetic division we get

1 -1 1 3 \ -1

-1 2 -3

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1 -2 3 0

$m^3 - m^2 + m + 3 = (m+1)(m^2 - 2n + 3) = 0$

$\Rightarrow \quad m = -1$ or $m^2 - 2n + 3 = 0 \quad \Rightarrow$

$(m-1)^2 = -2 \quad \Rightarrow \quad m = 1 \pm \sqrt{2}\,i$.

The general solution is

$y = C_1e^{-t} + e^t\left(C_2cos\sqrt{2}\,t + C_3sin\sqrt{3}\,t\right)$.

b) $y''' + 2y'' + 5y' - 26y = 0$

$m^3 + 2m^2 + 5m - 26 = 0$

The possible integer solutions are

$\pm$ 1, 2, 13, and 26 .

Try $m = 2$. ← graph $y = x^3 + 2x^2 + 5x - 26$ and you will see that $x = 2$ is a zero of the function

$$\begin{array}{llll} 1 & 2 & 5 & -26 \setminus 2 \\ & 2 & 8 & 26 \\ \hline 1 & 4 & 13 & 0 \end{array}$$

$m^3 + 2m^2 + 5m - 26 = 0 \quad \Rightarrow$

$(m-2)(m^2 + 4m + 13) = 0$

$\Rightarrow \quad m = 2$ or $(m+2)^2 = -13 + 4 \quad \Rightarrow$

$m = -2 \pm \sqrt{-9} = -2 \pm 3i\,.$

The general solution is

$y = C_1 e^{2t} + e^{-2t}(C_1 cos\,3t + C_2 sin\,3t)\,.$

c) $y^{(4)} + 13y'' + 36y = 0$

$m^4 + 13m^2 + 36 = 0 \quad \Rightarrow$

$(m^2 + 4)(m^2 + 9) = 0 \quad \Rightarrow$

$m = \pm 2i, \ \pm 3i\,.$

The general solution is

$y = C_1 sin\,2t + C_2 cos\,2t + C_3 sin\,3t + C_4 cos\,3t\,.$