Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 4.10 (p. 246)- Forced Mechanical Vibrations

3. $y'' + 9y = 2\,cos\,3t$ where $y(0) = 1$ and $y'(0) = 0$

$y_c = a\,cos\,3t + b\,sin\,3t$

$\boldsymbol{y_p = ktsin\,3t}$

$\boldsymbol{y_p' = ksin\,3t + 3ktcos\,3t}$

$\boldsymbol{y_p'' = 3kcos\,3t + 3kcos\,3t - 9ktsin\,3t}$

$(D^2 + 9D)y_p = 3kcos\,3t + 3kcos\,3t - 9ktsin\,3t + 9ktsin\,3t =$

$6kcos\,3t = 2cos\,3t \;\;\Rightarrow\;\; k \;=\; 1/3$

$y = a\,cos\,3t + b\,sin\,3t + \frac{1}{3}tsin\,3t$

$y(0) = 1 \; = a$

$y' = \; - 3acos\,3t - 3bsin\,3t + \frac{1}{3}sin\,3t + \frac{1}{3}tcos\,3t$

$y'(0) = \; - 3a = 0 \;\;\Rightarrow\;\; a = 0$

Ans: $\boldsymbol{y = cos\,3t \; + \frac{1}{3}tsin\,3t}$

9. An 8-kg mass is attached to a spring hanging from the ceiling, thereby causing the spring to spring to stretch $1.96\,m$ upon coming to rest at equilibrium. At time $t = 0$, an external force $F(t) = cos\,2t\,N$ is applied to the system. The damping constant for the system is 3 N-sec/m. Determine the steady state solution for the system.

$F = mg = 8 \times 9.807 = 78.4\,N$

$78.4 = 1.96\,k \;\;\Rightarrow\;\; k \;= 78.4/1.96 = 40$

$8y'' + 3y' + 40y = cos\,2t$

$y_p = a\,cos\,2t + b\,sin\,2t \;\;\Leftarrow\;\; \times 40$

$y_p' = \; - 2a\,sin\,2t + 2b\,cos\,2t \;\;\Leftarrow\;\; \times 3$

$y'' = \; - 4a\,cos\,2t - 4b\,sin\,2t \;\;\Leftarrow\;\; \times 8$

$(40a + 6b - 32a)cos\,2t \; + \; (40b - 6a - 32b)sin\,2t \; = cos\,2t$

$$8a + 6b = 1 \qquad -48a + 64b = 0$$
$$-6a + 8b = 0 \qquad 48a + 36b = 6$$
$$100b = 6 \;\Rightarrow\; b = 6/100 = .06$$
$$-6a + .48 = 0 \;\Rightarrow\; a = .48/6 = .08$$

$$y_p = .08\,cos\,2t + .06\,sin\,2t$$
$$\phi = tan^{-1}\frac{\text{coef of cos}}{\text{coef of sin}} = tan^{-1}\frac{.08}{.06} = tan^{-1}\frac{4}{3} = .927$$
$$\sqrt{(.08)^2 + (.06)^2} = .1$$

Ans: $\boldsymbol{y_p = .1\,sin(2t + .927)}$

13. $\frac{32}{32}y'' + 2y' + 5y = 3\,cos\,4t$

$$(D^2 + 2D + 5)(acos\,4t + bsin\,4t) = 3\,cos\,4t$$

$$y_p = a\,cos\,4t + b\,sin\,4t \quad \Leftarrow \times 5$$
$$y_p' = -4a\,sin\,4t + 4b\,cos\,4t \quad \Leftarrow \times 2$$
$$y_p'' = -16a\,cos\,4t - 16\,b\,sin\,4t \quad \Leftarrow \times 1$$

$$(-16a + 8b + 5a)cos\,4t + (-16b - 8a + 5b)sin\,4t = 3cos\,4t$$
$$(-11a + 8b)cos\,4t + (-8a - 11b)sin\,4t = 3cos\,4t$$

$$-11a + 8b = 3 \qquad -121a + 88a = 33$$
$$-8a - 11b = 0 \qquad -64a - 88a = 0$$
$$-185a = 33 \;\Rightarrow\; a = \tfrac{-33}{185}$$

$$-88a + 64b = 24$$
$$88a + 121b = 0$$
$$185b = 24 \;\Rightarrow\; b = \frac{24}{185}$$

Therefore $y_p = \frac{-33}{185}cos\,4t + \frac{24}{185}sin\,45$ or

$y_p = .221\,sin(4t + \phi)$ where ϕ is in quad. 4 and

$$\phi = -\,tan^{-1}\left(\frac{|\text{coef of cos}|}{\text{coef of sin}}\right) =$$

$$-\,tan^{-1}(\frac{33}{24}) = -\,.942$$

$$\boldsymbol{y_p = .221\, sin(4t - .942)}$$