Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 3.2 (p. 104)- Compartmental Analysis

1. $\frac{8L}{min} \Rightarrow \mid 100\,L,\ .5 \text{ kg of salt} \mid \Rightarrow \frac{8\,L}{min}$

$\frac{.05\,kg}{L}$

$$x(t) = \text{ amt. of salt in tank; } \; x(0) = .5$$

$$\frac{dx}{dt} = \text{ input rate } - \text{ output rate}$$

$$\frac{dx}{dt} = \frac{8L}{min} \cdot \frac{.05\,kg}{L} - \frac{8\,L}{min} \cdot \frac{x(t)}{100}$$

$$x' = .4 - \frac{2}{25}x \;\; \Rightarrow \;\; x' + \frac{2}{25}x = .4\,.$$

$$\text{I.F.} = e^{\frac{2}{25}t}\,.$$

The solution to DE is

$$xe^{\frac{2}{25}t} = \int 0.4e^{\frac{2}{25}t} = .4 \cdot \frac{25}{2}e^{\frac{2}{25}t} = 5e^{\frac{2}{25}t} + C$$

Therefore $x(t) = Ce^{-\frac{2}{25}t} + 5$ and

$$x(0) = 0.5 \;\; \Rightarrow \;\; 0.5 = C + 5 \;\; \Rightarrow \;\; C = -4.5.$$

$$\boldsymbol{x(t) = 5 - 4.5e^{-\frac{2}{25}t}}$$

$.02$ kg/L $\cdot\, 100\,L = \mathbf{2}$

$$2 = 5 - 4.5e^{-\frac{2}{25}t} \;\; \Rightarrow \;\; e^{-\frac{2}{25}t} = \frac{30}{45} \;\; \Rightarrow t = \frac{-\,ln\,(30/45)}{.08} =$$

5.07 minutes

3. Input rate: $\dfrac{6\,L}{min} \;\Rightarrow\; |\,200\,tot\,L,\ 1\,L\,acid| \;\Rightarrow\; \dfrac{8\,L}{min}$

200 L at start; note that .5% of 200 = 1 L

Output rate: $\dfrac{6\,L}{min} \cdot 20\% = 6/5$ and $\dfrac{8\,L}{min} \cdot \dfrac{x(t)}{200-2t} = \dfrac{4x}{100-t}$

$$\frac{dx}{dt} = \frac{6}{5} - \frac{4x}{100-t} \;\Rightarrow\; x' + \frac{4}{100-t}x = \frac{6}{5} \text{ a linear DE}$$

$$u(x) = e^{\int \frac{4}{100-t}dt} = e^{-4ln(100-t)} = e^{ln(100-t)^{-4}} = (100-t)^{-4}.$$

$$x \cdot (100-t)^{-4} = \frac{6}{5}\int (100-t)^{-4}\,dt = \frac{6(100-t)^{-3}}{15} =$$
$$\tfrac{2}{5}(100-t)^{-3} + C.$$

$$x(t) = \frac{2}{5}(100-t) + C(100-t)^4$$

$$x(0) = 1 \;\Rightarrow\; 1 = \tfrac{2}{5}\cdot 100 + C(100)^4 \;\Rightarrow$$
$$1 = 40 + C(100)^4 \;\Rightarrow\; C = -39/(100)^4 = \frac{-39}{10^8}.$$

$$\boldsymbol{x(t) = \frac{2}{5}(100-t) - \frac{39}{10^8}(100-t)^4}$$

$$\frac{x(t)}{2(100-t)} = .10 \;\Rightarrow$$
$$\frac{\boldsymbol{\frac{2}{5}(100-t) - \frac{39}{10^8}(100-t)^4}}{2(100-t)} = \frac{2}{10} - \frac{39(100-t)^3}{2(10)^8} = .1$$

$$\frac{39(100-t)^3}{2(10)^8} = \frac{1}{10} \;\Rightarrow\; (100-t)^3 = \frac{2\cdot 10^7}{39}$$

$100 - t = 80.04 \;\Rightarrow\; t = 100 - 80.04 =$ **19.96** minutes.

7. $\frac{3gal}{min} \Rightarrow |\,60\,g\,brime|\;\;(3\,\frac{gal}{min}) \Rightarrow \;|60\,water| \;\;\Rightarrow \frac{3\,gal}{min}$

Set up 2 separate DE's, one for 1st tank and one for 2nd.

Let $x(t)$ be the amt of salt in 1st tank.
Input of salt in tank 1 is 0 and the output of salt is
$3\,\frac{gal}{min}\cdot\frac{x}{60} = \frac{x}{20}\frac{gal}{min}$

$$\frac{dx}{dt} = -\frac{1}{20}x \;\;\Rightarrow\;\; \frac{dx}{x} = -\frac{1}{20}dt \;\;\Rightarrow\;\; ln\,x = -\frac{1}{20}t + K$$

$\boldsymbol{x = ke^{-\frac{1}{20}t}}$ where k is the initial amt of salt in tank 1. We will keep k. We don't need to find this value!

Now we set up a DE for tank 2 using $\boldsymbol{ke^{-\frac{1}{20}t}}$.
Therefore x is now the amt of salk in tank 2!
Input into tank 2 is $\frac{3\,gal}{min}\cdot\frac{\boldsymbol{ke^{-\frac{1}{20}t}}}{60} \boldsymbol{= \frac{1}{20}k\,e^{-t/20}}$.

The output from tank 2 is $\frac{3x}{60} = \frac{x}{20}$.

$$x' = \frac{1}{20}k\,e^{-t/20} - \frac{x}{20}$$

$$x' + \frac{1}{20}x = \frac{1}{20}k\,e^{-t/20}$$

$$u(x) = e^{\int \frac{1}{20}dt} = e^{t/20}$$

$$xe^{t/20} = \frac{1}{20}\int kdt = \frac{1}{20}kt + C$$

$x(t) = \frac{1}{20}kte^{-t/20} + Ce^{-t/20}$ where $x(0) = 0$.

$x(0) = Ce^{-t/20} = 0 \;\;\Rightarrow\;\; C = 0$ and

$$\boldsymbol{x(t) = \frac{1}{20} k t e^{-t/20}}$$

x will be a max. when x' is 0.

$$x' = \frac{1}{20} k e^{-t/20} - \frac{1}{400} k t e^{-t/20} = \frac{1}{20} k e^{-t/20} \left(1 - \frac{1}{20} t\right) = 0$$

when $1 = \frac{1}{20} t \quad \Rightarrow \quad t = 20$ minutes.

The max. amt of salt in tank 2 is $x(20) = \frac{1}{20} k(20) e^{-1} = k e^{-1}$.

The original amt of salt in tank 1 is k; therefore the ratio is

$$\frac{ke^{-1}}{k} = e^{-1} = .368 \quad \Rightarrow \text{the 2nd tanks 37\% as salty as tank 1.}$$

9. $\frac{dP}{dt} = kP \Leftarrow$ Malthusian Law where $P(0) = 1000$

$$\frac{dP}{P} = kdt \quad \Rightarrow \quad lnP = kt + C$$

$$P = e^{kt+C} = Ce^{kt}$$

$$P(0) = 1000 \quad \Rightarrow \quad C = 1000.$$

$$P(t) = 1000e^{kt} \quad \text{where } P(7) = 3000 = 1000e^{7k}.$$

$$e^{7k} = 3 \quad \Rightarrow \quad k = \frac{ln\,3}{7} = 0.156945\,.$$

$$P(30) = 1000e^{\frac{ln3}{7} * 30} = 110,868\,.$$