Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 2.6 (p. 78)- Substitutions and Bernoulli's Equation

11. Use the method of "Homogeneous Equations" to solve
$(y^2 - xy)dx + x^2 dy = 0$

The DE has homogeneous coef's.

Let $v = y/x \ or \ y = vx$
$(v^2x^2 - vx^2)dx + x^2 dy = 0$

$(v^2 - v)dx + dy = 0$
$v^2 dx + x dv = 0 \quad \Rightarrow \quad \dfrac{dx}{x} + \dfrac{dv}{v^2} = 0 \quad \Rightarrow$
$ln\,|x| + \dfrac{v^{-1}}{-1} = c \quad \Rightarrow ln\,|x| = x/y + c \quad \Rightarrow \quad \boldsymbol{y = \dfrac{x}{ln|x| + c}}$
and $y = 0$.

Remember that when using these techniques, we are rearranging the equation, and as a result, we can lose or gain some solutions.

15. Use the method of "Homogeneous Equations" to solve
$$y' = \frac{x^2 - y^2}{3xy}$$

Let $v = y/x \ or\, y = vx$
$dy/dx = \dfrac{x^2 - v^2x^2}{3vx^2} = \dfrac{1 - v^2}{3v}$

$3v(vdx + xdv) = (1 - v^2)dx$

$(4v^2 - 1)dx + 3xvdv = 0$

$\dfrac{dx}{x} + \dfrac{3v}{4v^2 - 1}dv = 0$

$ln(x) + \frac{3}{8} ln\big(4v^2 - 1\big) = ln\, c$
$x^{8/3}(4v^2 - 1) = c \;\;\Rightarrow\;\; x^{8/3}\Big(\frac{4y^2}{x^2} - 1\Big) = c \;\;\Rightarrow$
$x^{8/3}(4y^2 - x^2) = cx^2 \;\;\Rightarrow\;\; x^8(4y^2 - x^2)^3 = cx^6 \;\;\Rightarrow$
$\boldsymbol{x^2(4y^2 - x^2)^3 = c}$

21. Use Bernoulli's method to solve $y' + \dfrac{y}{x} = x^2y^2$.

Mult. the DE by y^{-2}.
$y^{-2}\dfrac{dy}{dx} + \dfrac{1}{x}y^{-1} = x^2$

Let $\boldsymbol{w = y^{-1}} \;\;\Rightarrow\;\; dw = -y^{-2}dy$

$-y^{-2}\dfrac{dy}{dx} - \dfrac{1}{x}w = -x^2$

$\dfrac{dw}{dx} - \dfrac{1}{x}w = -x^2 \;\;\Rightarrow\;\; u = e^{\int(-1/x)dx} = x^{-1}$

$wx^{-1} = \int(-x)dx = -x^2/2 + c \;\;\Rightarrow$

$\dfrac{1}{xy} = \dfrac{-x^2}{2} + c \;\;\Rightarrow\;\; \boldsymbol{y = \dfrac{2}{-x^3 + cx}}$ and $y = 0$.

22. Use Bernoulli's method to solve $y' - y = e^{2x}y^3$.

Mult. the DE by y^{-3}.
$y^{-3}dy/dx - y^{-2} = e^{2x}$

Let $\boldsymbol{w = y^{-2}} \;\;\Rightarrow\;\; dw/dx = -2y^{-3}dy/dx$
$-2y^{-3}dy/dx + 2w = -2e^{2x}$
$dw/dx + 2w = -2e^{2x} \;\;\Rightarrow\;\; u = e^{\int 2dx} = e^{2x}$

$$we^{2x} = -2\int e^{4x}dx = -\frac{e^{4x}}{2} + c$$

Ans: $\boldsymbol{y^{-2} = -\frac{e^{2x}}{2} + c\,e^{-2x}}$

26. Use Bernoulli's method to solve $\frac{dy}{dx} + y = e^x y^{-2}$.

Mult. the DE by y^2.

$$y^2\frac{dy}{dx} + y^3 = e^x$$

Let $\boldsymbol{w = y^3} \;\Rightarrow\; dw/dx = 3y^2 dy/dx$

$$3y^2\frac{dy}{dx} + 3y^3 = 3e^x$$

$$\frac{dw}{dx} + 3w = 3e^x \;\Rightarrow\; u = e^{\int 3dx} = e^{3x}$$

$$y^3e^{3x} = \int 3e^{4x}dx = \frac{3e^{4x}}{4} + c$$

Ans: $y^3 = \frac{3e^x}{4} + ce^{-3x}$

33. Solve the DE $(y - 4x - 1)^2 dx - dy = 0$.

This DE can be solved by a clever substitution.

Let $\boldsymbol{u = y - 4x - 1} \;\Rightarrow\; du = dy - 4dx$

$$u^2dx - (du + 4dx) = 0$$

$$\left(u^2 - 4\right)dx = du \;\Rightarrow\; dx = \frac{du}{u^2 - 4} = \frac{-1/4}{u+2} + \frac{1/4}{u-2}$$

$$4x + c = -ln|u + 2 + ln|u - 2 = ln\left|\left(\frac{u-2}{u+2}\right)\right|$$

$$4x + c = ln\left|\frac{u-2}{u+2}\right| \;\Rightarrow\; ce^{4x} = \frac{u-2}{u+2} \;\Rightarrow$$

$$ce^{4x}u + 2ce^{4x} = u - 2 \;\Rightarrow\; 2 + 2ce^{4x} = u(1 - ce^{4x})$$

$$u = \frac{2 + 2ce^{4x}}{1 - ce^{4x}} = y - 4x - 1 \;\Rightarrow\; y = 4x + 1 + \frac{2 + 2ce^{4x}}{1 - ce^{4x}}$$

$$y = 4x + \frac{1 - ce^{4x}}{1 - ce^{4x}} + \frac{2 + 2ce^{4x}}{1 - ce^{4x}} = \boldsymbol{4x + \frac{3 + c\,e^{4x}}{1 - c\,e^{4x}}}$$

40. Solve the DE $cos(x+y)dy = sin(x+y)dx$.

Rewriting the DE we get $dy/dx = tan(x+y)$.

This DE can be solved by a clever substitution.

Let $\boldsymbol{u = x + y} \;\Rightarrow\; du = dx + dy$

$$du - dx = tan\,u\,dx$$

$$du = (1 + tan\,u)dx \;\Rightarrow\; \frac{du}{1 + tan\,u} = dx$$

$$\frac{du}{1 + sin\,u/cos\,u} = dx \;\Rightarrow\; \frac{cos\,u}{cos\,u\ +\ sin\,u}du = dx$$

Mult. both sides by 2 and then add $-\ sin\,u +\ sin\,u$:

$$\frac{2\,cos\,u}{cos\,u\ +\ sin\,u} = 2dx$$

$$\frac{2\,cos\,u - sin\,u + sin\,u}{cos\,u\ +\ sin\,u}du = 2dx$$

$$\frac{cos\,u\ -\ sin\,u + cos\,u\ +\ sin\,u}{cos\,u\ +\ sin\,u}du = 2dx$$

$$\frac{cos\,u\ -\ sin\,u}{cos\,u + sin\,u}du + \frac{cos\,u + sin\,u}{cos\,u + sin\,u}du = 2dx$$

$$ln(cos\,u + sin\,u) + u = 2x + c$$

$$ln(cos\,u\ +\ sin\,u)\ + u = 2x + c \;\Rightarrow\; cos\,u\ +\ sin\,u = e^{2x-u+c}$$

$$cos(x+y) + sin(x+y) = ce^{2x-x-y}$$

Ans: $\boldsymbol{cos(x+y) + sin(x+y) = ce^{x-y}}$