Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 2.4 (p. 65)- Exact Differential Equations

3. Determine whether the DE is separable, linear, neither, or both.
$(ye^{xy} + 2x)dx + (xe^{xy} - 2y)dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^{xy} + yxe^{xy} \quad \Rightarrow \quad \text{the DE is exact.}$$

The DE is exact but it is not separable and not linear.
$(ye^{xy} + 2x)dx + (xe^{xy} - 2y)dy = 0 \;\; \Rightarrow$

$$(xe^{xy} - 2y)\frac{dy}{dx} + ye^{xy} + 2x = 0 \;\; \Rightarrow$$

$$\frac{dy}{dx} + \frac{ye^{xy}}{xe^{xy} - 2y} = -\frac{2x}{xe^{xy} - 2y}\,.$$

This DE is not linear for several reasons- for example the y on the right side. The 2nd term is no good either because of the e^{xy} in the numerator and the y in the denominator.

5. Determine whether the DE is separable, linear, neither, or both.
$y^2 dx + (2xy + cos\, y)dy = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2y \quad \Rightarrow \quad \text{DE is exact.}$$

This is tricky because it is linear if we make x the dependent variable.

$$\frac{dx}{dy} + \frac{2xy + cos\, y}{y^2} = 0$$

$\dfrac{dx}{dy} + \dfrac{2x}{y} = \dfrac{-\,cos\, y}{y^2} \quad \Rightarrow$ DE is linear with x as the dependent variable!

9. Find the general solution to the DE $(2xy+3)dx+(x^2-1)dy=0$ if it is exact.

The DE is exact because
$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}(2xy+3)=2x \text{ and } \frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(x^2-1\right)=2x.$$

Now we simply have to find $F(x,y)$. We do this by integrating the terms of the DE holding y constant if there is a dx and holding x constant if there is a dy.

$$\int 2xydx=y\int 2xdx=x^2y \text{ and } \int 3dx=3x \text{ and}$$
$$\int(-1)dy=-y.$$
Now $\int x^2dy$ also gives x^2y, but only include x^2y once.
$F(x,y)=x^2y+3x-y$ because
$$dF=(2xy+3)dx+\left(x^2-1\right)dy.$$

The solution to the DE is $\boldsymbol{x^2y+3x-y=C}$.

11. Solve $(\cos x\cos y+2x)dx-(\sin x\sin y+2y)dy=0$ if the DE is exact.

The DE is exact because $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-\cos x\sin y+2$.
$$\int \cos x\cos y\,dx=\cos y\int\cos x\,dx=\sin x\cos y,\ \int 2x\,dx=x^2,$$
$$\int(-2)y\,dy=-y^2, \text{ and } F(x,y)=\sin x\cos y+x^2-y^2$$
because $dF=(\cos x\cos y+2x)dx+(-\sin x\sin y-2y)dy$.

The solution to the DE is $\boldsymbol{\sin x\cos y+x^2-y^2=C}$.

15. Solve $cos\,\theta\,dr - (r\,sin\,\theta - e^{\theta})d\theta = 0$ if DE is exact.

$$\frac{\partial\,cos\,\theta}{\partial\,\theta} = -\,sin\,\theta \text{ and } \frac{\partial\,(-\,rsin\,\theta + e^{\theta})}{\partial r} = -\,sin\,\theta$$

$\Rightarrow$ DE is exact.

$$\int cos\,\theta dr = cos\,\theta \int dr = r\,cos\,\theta$$

$$\int(-\,r\,sin\,\theta)d\theta = -\,r\int sin\,\theta\,d\theta = r\,cos\,\theta \text{ (already have this)}$$

$$\int e^{\theta}d\theta = e^{\theta} \text{ and } F(x,y) = r\,cos\,\theta + e^{\theta} \text{ because}$$

$$dF(r,\theta) = cos\,\theta\,dr + (-\,r\,sin\,\theta + e^{\theta})d\theta\,.$$

The solution is $\boldsymbol{r\,cos\,\theta + e^{\theta} = C}$.

17. Solve the DE if it is exact.
$(1/y)dx - (3y - x/y^2)dy = 0$

$$\frac{\partial\,y^{-1}}{dy} = -\,y^{-2} \text{ and } \frac{\partial(-\,3y + \,xy^{-2})}{dx} = y^{-2}$$

The DE is not exact.

23. Solve $\left(e^t y + te^t y\right)dt + \left(te^t + 2\right)dy = 0\,;\; y(0) = -\,1$.

$$\frac{\partial M}{\partial y} = e^t + te^t \text{ and } \frac{\partial N}{\partial t} = e^t + te^t \quad \Rightarrow \quad \text{DE is exact.}$$

$\int e^t ydt = ye^t$ ← we don't need this because it would cancel with the 2nd integral. See the integral below.

$\int te^t ydt = y\int te^t dt = y\,te^t - ye^t$ ← the $-\,ye^t$ cancels

$$\int te^t dy = te^t \int dy = y\,te^t$$

$$\int 2d\,y = 2y$$

So $F(x,y)$ is simply $F(x,y) = y\,te^t + 2y$.

Check that $dF(x,y)$ gives the 5 terms on the left side of the DE.

The general solution to the DE is $y\,te^t + 2y = C$.

$y(0) = -1 \Rightarrow \quad 0 + (-2) = C \quad \Rightarrow \quad C = -2$.

The specific solution is $yte^t + 2y = -2$ or $\boldsymbol{y = \dfrac{-2}{te^t + 2}}$.