Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 2.3 (p. 54)- Linear Differential Equations

1. Is the DE separable, linear, neither, or both?

$$x^2\frac{dy}{dx} + sin\,x - y = 0 \quad\Rightarrow\quad x^2\frac{dy}{dx} - y = -\,sin\,x$$

DE is Linear because it has the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x).$$

It is not separable because there is no way we can get
$x^2dy + (sin\,x - y)dx = 0$ into the form
$a(y)dy + b(x)dx = 0$. The problem is with the $sin\,x - y$.

3. $$(t^2+1)\frac{dy}{dt} = yt - y \quad\Rightarrow\quad \left(t^2+1\right)dy = y(t-1)dt \quad\Rightarrow$$

$$\frac{dy}{y} = \frac{t-1}{t^2+1}dt \quad\Rightarrow$$ DE is separable.

$$(t^2+1)dy = y(t-1) \quad\Rightarrow\quad \frac{t^2+1}{t-1}\frac{dy}{dx} - y = 0 \quad\Rightarrow$$

DE is also linear.

So the DE is BOTH linear and separable.

5. $$x\frac{dx}{dt} + t^2x = sin\,t$$

Writing the DE as $xdx = (sin\,t - t^2x)dt$ we see that we can not rewrite the DE in the form $a(x)dx = b(t)dt$. Therefore the DE is not separable, and it is not linear because of the $x\frac{dx}{dt}$ where x is the dependent variable. Note that $a(t)dx$ is ok but not $a(x)dx$.

So the DE is NEITHER separable nor linear.

7. Find the general solution to the DE $\dfrac{dy}{dx} - y - e^{3x} = 0.$

Rewrite the DE as
$\dfrac{dy}{dx} - y = e^{3x} \quad \Rightarrow \quad u(x) = e^{-\int dx} = e^{-x}$

The solution to the DE has the form
$ye^{-x} = \displaystyle\int e^{2x} dx = e^{2x}/2 + C$ ← mult. by e^x
The solution is $\boldsymbol{y = \dfrac{e^{3x}}{2} + Ce^x}$.

16. $(x^2+1)\dfrac{dy}{dx} = x^2 + 2x - 1 - 4xy$

The DE is linear because we can rewrite it as
$y' + \dfrac{4xy}{x^2+1} = \dfrac{x^2+2x-1}{x^2+1} \quad \Rightarrow$
$u(x) = e^{\int \frac{4x}{x^2+1} dx} = e^{2ln(x^2+1)} = (x^2+1)^2$.
The solution is given by
$y(x^2+1)^2 = \displaystyle\int (x^2+1)(x^2+2x-1)dx =$
$\displaystyle\int (x^4 + 2x^3 + 2x - 1)dx$.
The solution is $\boldsymbol{y(x^2+1)^2 = \dfrac{x^5}{5} + \dfrac{x^4}{2} + x^2 - x + C}$.

17. Solve the initial value problem $\dfrac{dy}{dx} - \dfrac{y}{x} = xe^x;\ y(1) = e - 1$.

$y' - \dfrac{1}{x}y = xe^x$ is a linear DE and

$$u(x) = e^{\int \frac{-1}{x} dx} = e^{ln(x^{-1})} = \frac{1}{x} \;.$$

The solution is given by

$y/x = \int e^x dx = e^x + C \quad \Rightarrow \quad y = xe^x + Cx$

$y(1) = e - 1 \quad \Rightarrow \quad e - 1 = e + C \quad \Rightarrow \quad C = -1$

Ans: $\boldsymbol{y = xe^x - x}$

19. $t^3 \frac{dx}{dt} + 3t^2 x = t; \; x(2) = 0$

Rewrite the DE as

$\frac{dx}{dt} + \frac{3}{t} x = t^{-2} \quad \Rightarrow \quad u(t) = e^{\int \frac{3}{t} dt} = t^3$

$xt^3 = \int t dt = t^2/2 + C$

$x = \frac{1}{2t} + \frac{C}{t^3}$

$x(2) = 0 \quad \Rightarrow \quad 0 = \frac{1}{4} + \frac{C}{8} \quad \Rightarrow \quad -\frac{1}{4} = \frac{C}{8} \quad \Rightarrow \quad C = -2\,.$

Ans: $\boldsymbol{x = \frac{1}{2t} - \frac{2}{t^3}}$