Fundamentals of Differential Equations
by Nagle, Saff, and Snider (7th edition)

Section 2.2 (p. 46)- Separable Equations

12. Solve the DE $x\dfrac{dv}{dx} = \dfrac{1-4v^2}{3v}$.

The DE is separable.

$$x\frac{dv}{dx} = \frac{1-4v^2}{3v} \quad \Rightarrow \quad \frac{3v}{1-4v^2}dv = \frac{dx}{x} \quad \Rightarrow$$

the solution to the DE is given by the equation
$\displaystyle\int \frac{3v}{1-4v^2}dv = \int \frac{dx}{x} + C$.

Let $u = 1-4v^2 \quad \Rightarrow \quad du = -8vdv$

$$\int \frac{3v}{1-4v^2}dv = \frac{-3}{8}\int \frac{-8v}{1-4v^2}dv = \frac{-3}{8}\int \frac{du}{u}$$

$= -\dfrac{3}{8}ln|1-4v^2|$ and $\displaystyle\int \frac{dx}{x} = ln|x|$.

The solution to the DE is
$-\dfrac{3}{8}ln|1-4v^2| = ln|x| + C$ or better using $ln|C|$
in place of C,
$-\dfrac{3}{8}ln|1-4v^2| = ln|x| + ln|C| \quad \Rightarrow$
$-3ln|1-4v^2| = 8ln|x| + 8\,ln|C| \quad \Rightarrow$
$ln|1-4v^2|^{-3} = ln|Cx|^8 \quad \Rightarrow \quad |1-4v^2|^{-3} = |Cx^8|$
$\Rightarrow \quad (1-4v^2)^{-3} = Cx^8 \quad \Rightarrow \quad (1-4v^2)^3 = \dfrac{C}{x^8}$
$\Rightarrow \quad 1-4v^2 = \sqrt[3]{\dfrac{C}{x^8}} \quad \Rightarrow \quad 4v^2 = 1 - Cx^{-8/3}$
or if you want $\boldsymbol{4v^2 = 1 + Cx^{-8/3}}$.

15. Solve the DE $y^{-1}dy + ye^{cos\,x}sin\,xdx = 0$.

Divide the DE

$$y\,sin\,xe^{cos\,x}dx + y^{-1}dy = 0$$

by y and we get the DE $sin\,x\,e^{cos\,x}dx + y^{-2}dy = 0$.

The solution to the DE is given by

$$\int sin\,x\,e^{cos\,x}dx + \int y^{-2}dy + C = 0\,.$$

$$\int sin\,x\,e^{cos\,x}dx = -\int e^{u}du$$

(where $u = cos\,x,\ du = -sin\,x\,dx$)

$$= -e^{u} = -e^{cos\,x}.$$

$$\int y^{-2}dy = \frac{y^{-1}}{-1} = -\frac{1}{y}\,.$$

Therefore the solution to the DE is

$$-e^{cos\,x} - \frac{1}{y} + C = 0 \;\Rightarrow\; C - e^{cos\,x} = 1/y \;\Rightarrow$$

$$\boldsymbol{y = \frac{1}{C - e^{cos\,x}}}\,.$$

19. Solve the initial value problem

$$\frac{dy}{dx} = 2\sqrt{y+1}\,cos\,x;\; y(\pi) = 0\,.$$

The solution to the DE is given by

$$\int \frac{dy}{2\sqrt{y+1}} = \int cos\,x\,dx + C\,.$$

$$\int \frac{dy}{2\sqrt{y+1}} = cos\,x\,dx \quad \Leftarrow \quad u = y+1;\; du = dy$$

$$= \int \frac{du}{2u^{1/2}} = \frac{u^{1/2}}{2(1/2)} = \sqrt{y+1} \text{ and } \int cos\,x\,dx \;=\; sin\,x\ .$$

The general solution to the DE is

$$\sqrt{y+1} = sin\,x + C \;\Rightarrow\; sin\,x \;=\; \sqrt{y+1} + C\ .$$

Since $y(\pi) = 0$,

$$sin\,\pi = 1 + C \;\Rightarrow\; C = -1$$

The specific solution is $sin\,x = -1 + \sqrt{y+1} \;\Rightarrow$

$$\sqrt{y+1} = sin\,x + 1 \;\Rightarrow\; y + 1 = (sin\,x + 1)^2 \;\Rightarrow$$

$$\boldsymbol{y = sin^2 x + 2\,sin\,x}\ .$$

24. Solve the initial value problem

$$\frac{dy}{dx} = 8x^3 e^{-2y};\ y(1) = 0\ .$$

The solution is given by

$$\int e^{2y} dy = \int 8x^3 dx + C \;\Rightarrow$$

$$\frac{e^{2y}}{2} = 2x^4 + C \;\Rightarrow\; e^{2y} = 4x^4 + C\ .$$

Since $y(1) = 0$,

$1 = 4 + C \;\Rightarrow\; C = -3$ and the specific solution is

$$e^{2y} = 4x^4 - 3 \;\Rightarrow\; 2y = ln(4x^4 - 3) \;\Rightarrow$$

$$\boldsymbol{y = \tfrac{1}{2} ln(4x^4 - 3)}\ .$$

34. The Thermometer Problem.

a) Solve the DE $\frac{dT}{dt} = k(T - M)$ ← I use $T - M$ because $T > M$ in part b.

$$\frac{dT}{T-M} = kdt \;\; \Rightarrow \;\; ln|T-M| = kt + C \;\; \Rightarrow$$

$ln(T-M) = kt + C \;\; \Rightarrow \;\; T - M = e^{kt+C}$ or better Ce^{kt}

$\Rightarrow \;\; T - M = Ce^{kt} \;\; \Rightarrow \;\; \boldsymbol{T = M + Ce^{kt}}$.

b) At time 0, $T = 100$ and $M = 70°$.

$T = M + Ce^{kt} \;\; \Rightarrow \;\; 100 = 70 + C \;\; \Rightarrow \;\; C = 30.$

Therefore $T = 70 + 30e^{kt}$.

Since after 6 minutes, $T = 80°$,
$80 = 70 + 30e^{k6} \;\; \Rightarrow \;\; 30e^{6k} = 10 \;\; \Rightarrow$

$6k = ln(1/3) \;\; \Rightarrow \;\; 6k = -ln(3) \;\; \Rightarrow \;\; k = -\frac{ln\,3}{6}$.

$T = 70 + 30e^{-t ln(3)/6} = 70 + 30(e^{ln\,3})^{-t/6} \;\; \Rightarrow$

$T = 70 + 30 \cdot 3^{-t/6}$.

After 20 minutes, $T = 70 + 30\left(3^{-20/6}\right) = \mathbf{70.77°}$.