Solutions (Practice Test 2)
(Revised April 8, 2009)

A 1) Homogeneous coefficients

$y^2dx - x(2x+3y)dy = 0$

Let $x = uy$ or $u = x/y$

$y^2dx = (2u^2y^2 + 3uy^2)dy \;\Rightarrow\; (udy + ydu)1 = (2u^2+3u)dy$

$\Rightarrow\; ydu = \left(2u^2+2u\right)dy \;\Rightarrow\; \dfrac{du}{u(u+1)} = \dfrac{2}{y}dy \;\Rightarrow$

$\dfrac{du}{u} + \dfrac{-1du}{u+1} = \dfrac{2dy}{y} \;\Rightarrow\; ln\, u - ln(u+1) = ln\, y^2 + ln\, c \;\Rightarrow$

$\dfrac{u}{u+1} = cy^2 \;\Rightarrow\; \dfrac{x/y}{x/y+1} = cy^2 \;\Rightarrow\; \dfrac{x}{y+x} = cy^2 \;\Rightarrow$

$$y^2(x+y) = cx$$

2) Homogenous coeff's and Bernoulli's eq

$y(x+3y)dx + x^2dy = 0$

$y = vx \;\; or\, v = \dfrac{y}{x}$

$(vx^2 + 3v^2x^2)dx + x^2dy = 0 \;\Rightarrow$

$(v+3v^2)dx + vdx + xdv = 0 \;\Rightarrow$

$v(3v+2)dx + xdv = 0 \;\Rightarrow\; \dfrac{dx}{x} + \dfrac{dv}{v(3v+2)} = 0 \;\Rightarrow$

$\dfrac{dx}{x} + \dfrac{\frac{1}{2}dv}{v} + \dfrac{-\frac{3}{2}dv}{3v+2} = 0 \;\Rightarrow\; \dfrac{2\,dx}{x} + \dfrac{dv}{v} - \dfrac{3dv}{3v+2} = 0$

$\Rightarrow\; ln\, x^2 + ln\, v - ln(3v+2) = c \;\Rightarrow$

$x^2 = \dfrac{c(3v+2)}{v} = \dfrac{c(3y+2x)}{y} \;\Rightarrow\; yx^2 = c(3y+2x)$

Alternative Solution

$\dfrac{xy+3y^2}{x^2} + \dfrac{dy}{dx} = 0$

$y^{-2}y' + \dfrac{1}{x}y^{-1} = -\dfrac{3}{x^2} \;\Rightarrow\; v = y^{-1}$ and $\dfrac{dv}{dx} = -y^{-2}\dfrac{dy}{dx}$

$$\frac{dv}{dx} - \frac{1}{x}v = \frac{3}{x^2} \quad \Rightarrow \quad u(x) = e^{\int \frac{-1}{x}dx} = x^{-1}$$

$$x^{-1}y^{-1} = \int \frac{3}{x^3}dx = \frac{-3}{2}x^{-2} + c \quad \Rightarrow$$

$$\frac{1}{xy} = \frac{-3}{2x^2} + c \quad \Rightarrow \quad \boldsymbol{2x^2y}\left(\frac{1}{xy}\right) = \boldsymbol{2x^2y}\left(\frac{-3}{2x^2} + c\right) \quad \Rightarrow$$

$$2x = -3y + 2cx^2y \quad \Rightarrow \quad 2x + 3y = cx^2y$$

3) Bernoulli's Equation

$$6x^2dy - y(x + 2y^3)dx = 0$$

$$y' - \frac{1}{6x}y = \frac{y^4}{3x^2} \quad \Rightarrow \quad y^{-4}y' - \frac{1}{6x}y^{-3} = \frac{1}{3x^2}$$

$$v = y^{-3} \quad \Rightarrow \quad \frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$$

$$-3y^{-4}\frac{dy}{dx} + \frac{1}{2x}y^{-3} = -\frac{1}{x^2} \quad \Rightarrow \quad \frac{dv}{dx} + \frac{1}{2x}v = \frac{-1}{x^2} \quad \Rightarrow$$

$$u(x) = e^{\int \frac{1}{2x}dx} = x^{1/2}$$

$$y^{-3}x^{1/2} = \int \frac{-1}{x^{3/2}}dx = 2x^{-1/2} + c \ \Leftarrow mult\,by\,x^{1/2}$$

$$xy^{-3} = 2 + cx^{1/2} \quad \Rightarrow \quad x = 2y^3 + cy^3x^{1/2}$$

$$x - 2y^3 = cy^3x^{1/2} \quad \Rightarrow \quad (x - 2y^3)^2 = cy^6x$$

4) Homogeneous Coefficients

$$(3x^2y + 2xy^2)dx + (4x^3 + 5x^2y)dy = 0$$

Let $y = wx \quad \Rightarrow \quad dy = wdx + xdw \quad \Rightarrow \quad w = \dfrac{y}{x}$.

$$(3x^2wx + 2xw^2x^2)dx + (4x^3 + 5x^2wx)dx = 0$$

Factor out x^3 and replace for dy.

$$(3w + 2w^2)dx + (4 + 5w)(wdx + xdw) = 0$$

$$(3w + 2w^2 + [4w + 5w^2])dx + (4 + 5w)xdw = 0$$

$$(7w + 7w^2)dx + (4 + 5w)xdw = 0$$

$$\frac{7dx}{x} + \frac{4 + 5w}{w(w+1)}dw = 0$$

$$\frac{4+5w}{w(w+1)} = \frac{a}{w} + \frac{b}{w+1}$$
$$4+5w = a(w+1)+bw$$
$$w=0 \;\Rightarrow\; 4=a$$
$$w=-1 \;\Rightarrow\; -1=-b \;\Rightarrow\; b=1$$
$$\int \frac{7dx}{x} + \int\left(\frac{4}{w}+\frac{1}{w+1}\right)dw = ln\, C$$

$$7\,ln\, x + 4\,ln\, w + ln(w+1) = ln\, C$$
$$ln\, x^7 w^4(w+1) = ln\, C \;\Rightarrow\; x^7 w^4 (w+1) = C$$
$$\Rightarrow\; x^7 \frac{y^4}{x^4}\left(\frac{y}{x}+1\right) = C \;\Rightarrow\; x^3 y^4\left(\frac{y+x}{x}\right) = C$$
$$\Rightarrow\; x^2y^4(x+y) = C\,.$$

An alternative method- using a spectial integrating factor.

$$\boldsymbol{\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)}$$

$$(3x^2y+2xy^2)dx + (4x^3+5x^2y)dy = 0$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \left(3x^2+4xy\right) - \left(12x^2+10xy\right) = -9x^2-6xy.$$

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-9x^2-6xy}{3x^2y+2xy^2} = \frac{-3x(3x+2y)}{xy(3x+2y)} = \frac{-3}{y}$$

The integr. factor is $e^{-\int\left(\frac{-3}{y}\right)dy} = e^{\int \frac{3}{y}dy} = e^{3lny}$ $= y^3$.

Mult. the DE by y^3 and we get the exact DE

$(3x^2y^4+2xy^5)dx + (4x^3y^3+5x^2y^4)dy = 0$

$F(x,y) = x^3y^4 + x^2y^5$ and the solution is

$x^3y^4 + x^2y^5 = C$ or $x^2y^4(x+y) = C$.

B 1. $m^2 - m - 2 = 0 \;\Rightarrow\; (m-2)(m+1) = 0$

$\Rightarrow\; m = 2$ or -1

Ans: $y = c_1e^{2x} + c_2e^{-x}$

2. $m^2 + 5m + 6 = 0 \quad \Rightarrow \quad (m+3)(m+2) = 0$

$\Rightarrow \quad m = -3$ or -2

Ans: $y = c_1 e^{-3x} + c_2 e^{-2x}$

3. $m^2 + 10m + 25 = 0 \quad \Rightarrow \quad (m+5)^2 = 0 \quad \Rightarrow$

$m = -5$ is a root of mult. 2

Ans: $y = c_1 e^{-5x} + c_2 x e^{-5x}$

4. $m = -1 \pm i\sqrt{3}$

Ans: $y = e^{-x}\left(c_1 cos\sqrt{3}\,x + c_2 sin\sqrt{3}x\right)$

5. $m = 1 \pm \sqrt{6}$

Ans: $y = c_1 e^{\left(1+\sqrt{6}\right)x} + c_2 e^{\left(1-\sqrt{6}\right)x}$

6. $y_p = asin\,5x$

$y_p'' = -25asin\,5x$

$(D^2 + 16)y_p = -25asin\,5x + 16asin\,5x = -9asin\,5x$

$= sin\,5x \quad \Rightarrow \quad a = -1/9$

Ans: $y = -\dfrac{1}{9}sin\,5x + c_1 cos\,4x + c_2 sin\,4x$

C. 1. $y = Ae^{-4x} + Be^{3x} \quad \Rightarrow \quad y(0) = A + B = 1$

$y' = -4Ae^{-4x} + 3Be^{3x} \quad \Rightarrow \quad y'(0) = -4A + 3B = 0$

$-3A - 3B = -3$

$-4x + 3B = 0$

$-7A = -3 \quad \Rightarrow \quad A = 3/7$ and $B = 4/7$

Ans: $y = \dfrac{3}{7}e^{-4x} + \dfrac{4}{7}e^{3x}$

2. $y = Ae^x + Bxe^x$

$y' = Ae^x + Be^x + Bxe^x = (A+B)e^x + Bxe^x$

$y(0) = 0 = A; \; y'(0) = 2 = A + B \quad \Rightarrow \quad B = 2$

Ans: $y = 2xe^x$

3. $y_p = ke^{2x};\; y_p'' = 4ke^{2x}$

$(D^2-1)y_p = 4ke^{2x} - ke^{2x} = 3ke^{2x} = e^x \;\Rightarrow\; k = 1/3$

$y = Ae^x + Be^{-x} + \frac{1}{3}e^{2x} \;\Rightarrow\; y(0) = A + B + 1/3 = 0$

$y' = Ae^x - Be^{-x} + \frac{2}{3}e^{2x} \;\Rightarrow\; y'(0) = A - B + 2/3 = 0$

$A + B = -1/3$

$A - B = -2/3$

$2A = -1 \;\Rightarrow\; A = -1/2$ and $-\frac{1}{2} + B = -\frac{1}{3} \;\Rightarrow$
$B = \frac{1}{6}$.

Ans: $y = -\frac{1}{2}e^x + \frac{1}{6}e^{-x} + \frac{1}{3}e^{2x}$

D. 1. $y_p = kx\cos 5x$

$y_p' = k\cos 5x - 5kx\sin 5x$

$y_p'' = -10k\sin 5x - 25kx\cos 5x$

$(D^2+25)y_p = -10k\sin 5x - 25kx\cos 5x + 25kx\cos 5x$

$= -10k\sin 5x = \sin 5x \;\Rightarrow\; k = -\frac{1}{10}$

Ans: $y = -\frac{1}{10}x\cos 5x + c_1\sin 5x + c_2\cos 5x$

2. $y_p = kxe^{3x}$

$y_p' = ke^{3x} + 3kxe^{3x}$

$y_p'' = 3ke^{3x} + 3ke^{3x} + 9kxe^{3x} = 6ke^{3x} + 9kxe^{3x}$

$(D^2-9)y_p = 6ke^{3x} + 9kxe^{3x} - 9kxe^{3x} = 6ke^{3x} = e^{3x}$

$\Rightarrow\; k = 1/6$

Ans: $y = c_1e^{3x} + c_2e^{-3x} + \frac{1}{6}xe^{3x}$

3. $y_c = c_1e^{-2x} + c_2e^{-x}$

$y_p = a + bx;\; y_p' = b;\; y_p'' = 0$

$(D^2+3D+2)y_p = 0 + 3b + 2a + 2bx =$

$(2a+3b) + (2b)x = 1 + x \;\Rightarrow\; 2b = 1 \;\Rightarrow\; b = \frac{1}{2}$

$\Rightarrow\; 2a + 3b = 2a + \frac{3}{2} = 1 \;\Rightarrow\; 2a = -\frac{1}{2} \;\Rightarrow\; a = -\frac{1}{4}$

Ans: $y = -\frac{1}{4} + \frac{1}{2}x + c_1e^{-2x} + c_2e^{-x}$

4. $y_c = c_1 + c_2 e^{4x}$
$y_p = kx;\ y_p' = k;\ y_p'' = 0$
$\left(D^2 - 4D\right) y_p = 0 - 4k = 10 \ \Rightarrow \ k = -\frac{5}{2}$
Ans: $y = c_1 + c_2 e^{4x} - \frac{5}{2} x$

5. $y_c = c_1 e^{-3x} + c_2 e^{-x}$
$y_p = a sin\, 3x + b cos\, 3x \ \Leftarrow \times 3$
$y_p' = 3a cos\, 3x - 3b sin\, 3x \ \Leftarrow \times 4$
$y_p'' = -9a sin\, 3x - 9b cos\, 3x \ \Leftarrow \times 1$
$(D^2 + 4D + 3) y_p =$
$(-9a - 12b + 3a) sin\, 3x + (-9b + 12a + 3b) cos\, 3x$
$(-6a - 12b) sin\, 3x + (12a - 6b) cos\, 3x = sin\, 3x$

Therefore $-6a - 12b = 1$ and $12a - 6b = 0$
$-6a - 12b = 1$
$6a - 3b = 0$

$-15b = 1 \ \Rightarrow \ b = -\frac{1}{15}$
$a = \frac{1}{2} b \ \Rightarrow \ a = -\frac{1}{30}$
Ans: $y = c_1 e^{-3x} + c_2 e^{-x} - \frac{1}{30} sin\, 3x - \frac{1}{15} cos\, 3x$

E. W$[cos\, 3x,\ sin\, 3x] = \begin{vmatrix} cos\, 3x & sin\, 3x \\ -3 sin\, 3x & 3 cos\, 3x \end{vmatrix} =$
$3cos^2 3x + 3 sin^2 3x = 3(1) = 3 \neq 0$.
Therefore the functions are linearly independent.

F. Let y be an arbitrary function where the necessary derivatives exist.
$D(xD - 1)y = D(xDy - y) = D(xy' - y) = D(xy') - y' =$
$1y' + xy'' - y' = xy'' = xD^2 y$.
Therefore the 2 operators $D(xD - 1)$ and xD^2 are equal.

G. $m^2 + 2m + 5 = 0 \ \Rightarrow \ (m+1)^2 = -4 \ \Rightarrow \ m = -1 \pm 2i$
$y_c = e^{-x}(a cos\, 2x + b sin\, 2x)$

$y_p = ke^{-x}$
$(D^2 + 2D + 5)ke^{-x} = ke^{-x} - 2ke^{-x} + 5ke^{-x} = 4ke^{-x}$
$\quad = 8e^{-x} \quad \Rightarrow \quad k = 2$

$y = e^{-x}(a\cos 2x \,+\, b\sin 2x) + 2e^{-x}$
$y' = -e^{-x}(a\cos 2x + b\sin 2x) +$
$\quad e^{-x}(-2a\sin 2x + 2b\cos 2x) - 2e^{-x}$
$y(0) = 0 \;\Rightarrow\; a + 2 = 0 \;\Rightarrow\; a = -2$
$y'(0) = 8 \;\Rightarrow\; -1a + 2b - 2 = 8 \;\Rightarrow\; 2 + 2b - 2 = 8 \;\Rightarrow\; b = 4$

Ans: $y = e^{-x}(4\sin 2x - 2\cos 2x) + 2e^{-x}$

H. The aux. equation is $m^2 - 2m - 3 = 0 \;\Rightarrow\; (m-3)(m+1) = 0$
$\quad \Rightarrow \; m = 3,\, m = -1.$
$y_c = C_1e^{3x} + C_2e^{-x}$
$y_p = ax^2 + bx + c \;\Leftarrow \times -3$
$y_p' = 2ax + b \;\Leftarrow \times -2$
$y_p'' = 2a$
$(D^2 - 2D - 3)y_p = 2a - 4ax - 2b - 3ax^2 - 3bx - 3c$
$\quad = -3ax^2 + (-4a - 3b)x + (2a - 2b - 3c) = 3x^2 - 5$

$-3a = 3 \;\Rightarrow\; a = -1; \quad -4a - 3b = 0 \;\Rightarrow\; -3b = -4$
$\quad \Rightarrow \; b = 4/3$
$2a - 2b - 3c = -5$
$-2 - \dfrac{8}{3} - 3c = -5 \;\Rightarrow\; \dfrac{1}{3} = 3c \;\Rightarrow\; c = \dfrac{1}{9}$
Ans: $y = c_1e^{3x} + c_2e^{-x} - x^2 + \dfrac{4}{3}x + \dfrac{1}{9}$

I. Note that $12\cos^2 x = 12\dfrac{1 + \cos 2x}{2} = 6(1 + \cos 2x)$
$(D^2 + 1)y = 6 \,+\, 6\cos 2x$
$(D^2 + 1)y = 6$ has $y_p = 6$
$(D^2 + 1)y = \cos 2x$ has sol. of form $y_p = k\cos 2x$
$(D^2 + 1)y_p = -4k\cos 2x + k\cos 2x = -3k\cos 2x = 6\cos 2x$
$\quad \Rightarrow \; k = -2$
Ans: $y = c_1\cos x + c_2\sin x + 6 - 2\cos 2x$

J. 8L/min $\Rightarrow$ TANK $\Rightarrow$ 10L/min

Let $x(t)$ be the volume of acid in tank at any time t.

At time t, the volume in tank is $180-2t$

input rate $=$ 10% $\times$ 8L/min $= (.1)(8) = 0.8$

output rate $= \dfrac{x}{180-2t} \times 10 = \dfrac{10x}{180-2t} = \dfrac{5x}{90-t}$

$$\frac{dx}{dt} = input\,rate - output\,rate$$

$$\frac{dx}{dt} = 0.8 - \frac{5x}{90-t}$$

$$\boldsymbol{\frac{dx}{dt} + \frac{5x}{90-t} = \frac{8}{10}} \text{ where } \boldsymbol{x(0) = (.06)(180) = 10.8}$$

This DE is linear and it has an integrating factor of

$$e^{\int \frac{5\,dt}{90-t}} = e^{-5\,ln(90-t)} = (90-t)^{-5}$$

The solution to DE is

$$x(90-t)^{-5} = \frac{4}{5}\int (90-t)^{-5}dt = \frac{-4}{5}\int u^{-5}du$$

$$\frac{1}{5}(90-t)^{-4} + C \quad \Rightarrow$$

$$x = \frac{1}{5}(90-t) + C(90-t)^5$$

$$x(0) = 18 + 90^5 C = 10.8 \quad \Rightarrow \quad C = -\frac{7.2}{90^5}$$

$$x(t) = \frac{1}{5}(90-t) - \frac{7.2}{90^5}(90-t)^5$$