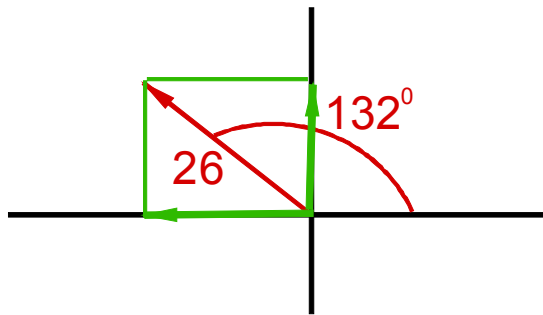


Math 242- Practice Test 1

1. Find the magnitude of the vector $v = 5i - 7j$.

Ans: The magnitude is $\sqrt{25 + 49} = \sqrt{74}$.

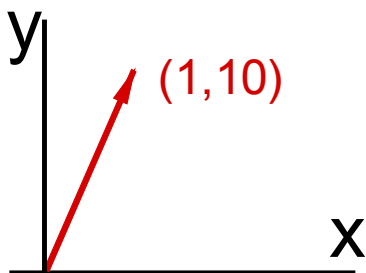
2. If a vector makes an angle of 132° with the x-axis and has magnitude 26, write this vector in the form $ai + bj$.



Ans: The vector v has the form $ai + bj$ where
 $v = 26 \cos 132^\circ i + 26 \sin 132^\circ j = -17.40i + 19.32j$.

3. Find a unit vector parallel to the graph of $y = x^3 - 2x - 1$ at the point $(2, 3)$.

Ans: $y' = 3x^2 - 2 \Rightarrow y'(2) = 12 - 2 = 10 \Rightarrow$ the vector parallel to the graph has slope 10. A vector with this direction (and slope) is $(1, 10)$.



A unit vector with the same direction as $(1, 10)$ is

$$\frac{1}{\sqrt{1+100}}(1, 10) = \left(\frac{1}{\sqrt{101}}, \frac{10}{\sqrt{101}} \right).$$

4. Find the standard equation of the sphere which has a diameter with endpoints $(3, 4, 0)$ and $(1, 2, -4)$.

Ans: The diameter has length $\sqrt{(3-1)^2 + (4-2)^2 + (0+4)^2} = \sqrt{4+4+16} = \sqrt{24}$. The radius of the sphere is $\frac{\sqrt{24}}{2} = \sqrt{6}$.

The center of the sphere is $\left(\frac{3+1}{2}, \frac{4+2}{2}, \frac{0-4}{2} \right) = (2, 3, -2)$.

The equation of this sphere is $(x-2)^2 + (y-3)^2 + (z+2)^2 = 6$.

5. If $v = 3i - 2j + 4k$ and $w = -i + 3j - 2k$ find the following if possible.
 a) $v \times w$ b) $(v \cdot w)(v + w)$ c) $v \times (w \cdot w)$

Ans: a) $v \times w = \begin{vmatrix} i & j & k \\ 3 & -2 & 4 \\ -1 & 3 & -2 \end{vmatrix} = i(4-12) - j(-6+4) + k(9-2) = -8i + 2j + 7k$.

b) $v \cdot w = (3, -2, 4) \cdot (-1, 3, -2) = -3 - 6 - 8 = -17$.

$(v \cdot w)(v + w) = -17(2, 1, 2) = (-34, -17, -34) = -34i - 17j - 34k$.

c) $v \times (w \cdot w) = v \times (1 + 9 + 4) = v \times 14$ is undefined.

6. Find the center and radius of the sphere whose equation is

$$x^2 - 8x + y^2 + 10y + z^2 + 6z = -5.$$

$$\text{Ans: } x^2 - 8x + 16 + y^2 + 10y + 25 + z^2 + 6z + 9 = -5 + 16 + 25 + 9 \Rightarrow$$

$$(x - 4)^2 + (y + 5)^2 + (z + 3)^2 = 45.$$

The center is $(4, -5, -3)$ and the radius is $\sqrt{45}$.

7. Find the angle θ between the vectors $u = (-3, 2, 1)$ and $v = (3, -1, -1)$.

$$\text{Ans: } \cos \theta = \frac{(-3, 2, 1) \cdot (3, -1, -1)}{\sqrt{9 + 4 + 1}\sqrt{9 + 1 + 1}} = \frac{-9 - 2 - 1}{\sqrt{14}\sqrt{11}} = \frac{-12}{\sqrt{154}} = -.96699 \text{ and } \theta = \cos^{-1}(-.96699) = 165.24^\circ.$$

8. Find the angle the vector $u = 2i - 2j + 3k$ makes with the positive x axis.

$$\text{Ans: If this angle is } \theta \text{ then } \cos \theta = \frac{2}{\sqrt{4 + 4 + 9}} = \frac{2}{\sqrt{17}}.$$

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{17}}\right) = 60.98^\circ.$$

9. Find the area of the parallelogram that has vectors $v = 2i + j - 3k$ and $w = i + 3j + 2k$ as adjacent sides.

Ans:

$$v \times w = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = (2 + 9)i - (4 + 3)j + (6 - 1)k =$$

$$11i - 7j + 5k.$$

$$\text{The area of the parallelogram is } \|v \times w\| = \sqrt{11^2 + (-7)^2 + 5^2} = \sqrt{195}.$$

10. Find the volume of the parallelepiped which has vectors

$(1, 2, 3)$, $(2, 3, -1)$, and $(-1, 1, 0)$ as adjacent sides.

$$\begin{aligned} u \cdot (v \times w) &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ -1 & 1 & 0 \end{vmatrix} = \\ &= 1(0 + 1) - 2(0 - 1) + 3(2 + 3) = \\ &= 1 + 2 + 15 = 18. \text{ The volume is } 18. \end{aligned}$$

11. Find sets of (a) parametric equations and (b) symmetric equations of the line through the 2 points $(1, 2, 3)$ and $(4, 6, 8)$.

Ans: A vector parallel to the line is $(4 - 1, 6 - 2, 8 - 3) = (3, 4, 5)$. These are the direction numbers for the line.

Using the point $(1, 2, 3)$, the parametric equations for the line are $x - 1 = 3t$, $y - 2 = 4t$, $z - 3 = 5t$ or

$$x = 3t + 1, \quad y = 4t + 2, \quad z = 5t + 3.$$

The symmetric equations are

$$\frac{x - 1}{3} = \frac{y - 2}{4} = \frac{z - 3}{5}.$$

12. Find an equation of the plane passing through the point $(0, 4, 0)$ perpendicular to the line $x = 1 - 2t$, $y = 3 + t$, $z = 2 + 2t$.

Ans: A vector perpendicular to the plane is $(-2, 1, 2)$.

Therefore the equation of the plane is given by

$$-2x + y + 2z = k \text{ where } (0, 4, 0) \text{ satisfies the equation.}$$

$$\text{This means } 0 + 4 + 0 = k \Rightarrow k = 4.$$

$$\text{The equation of the plane is } -2x + y + 2z = 4.$$

13. Find the equation of the plane which contains the points $(2, 0, 3)$, $(1, 1, 0)$, and $(3, 2, -1)$.

Ans: Two vectors which lie on the plane are

$v = (2 - 1, 0 - 1, 3 - 0) = (1, -1, 3)$ and

$w = (3 - 2, 2 - 0, -1 - 3) = (1, 2, -4)$.

$v \times w = (-2, 7, 3)$ is normal to the plane.

The equation of the plane has the equation

$$-2x + 7y + 3z = k \text{ and using } (1, 1, 0)$$

we get that $-2 + 7 = k \Rightarrow k = 5$.

The equation is $-2x + 7y + 3z = 5$.

You could also write the equation as

$$-2(x - 1) + 7(y - 1) + 3(z - 0) = 0.$$

14. Find the parametric equations and the symmetric equations of the line passing through the points $P(1, 2, -1)$ and $Q(5, -3, 4)$.

Ans:

A vector that is parallel to the line is \overrightarrow{PQ} is

$$(5 - 1, -3 - 2, 4 + 1) = (4, -5, 5).$$

Using the 1st point, the equations are

$$x - 1 = 4t, \quad y - 2 = -5t, \quad z + 1 = 5t$$

The symmetric equations are $\frac{x - 1}{4} = \frac{y - 2}{-5} = \frac{z + 1}{5}$.

15. Find the parametric equations of the line of intersection of the planes $x + 2y + z = 3$ and $x - 4y + 3z = 5$.

Ans:

$$x = -2y - z + 3 \text{ and } x = 4y - 3z + 5 \Rightarrow$$

$$-2y - z + 3 = 4y - 3z + 5 \Rightarrow 2z = 6y + 2$$

$$\Rightarrow z = 3y + 1. \text{ Let } y = t \Rightarrow z = 3t + 1.$$

Using 1st equation, we get

$$x + 2t + 3t + 1 = 3 \Rightarrow x = -5t + 2.$$

The parametric equations are $x = -5t + 2, y = t, z = 3t + 1$.

16. What kind of quadric surface is the graph of the equation. What

is the axis of the graph?

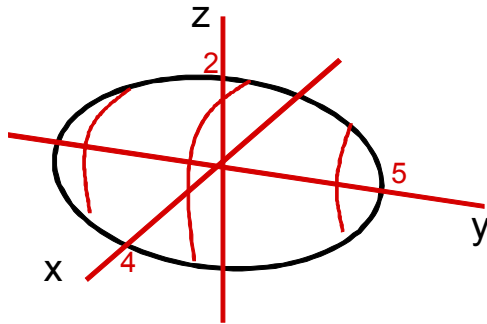
- a) $4x^2 - y^2 + z^2 = 0$ b) $-4x^2 + 16y^2 + z^2 = 64$
c) $4x^2 - 16y^2 - z^2 = 16$

- Ans: a) This equation is an elliptic cone. The axis of the cone is the y axis.
b) This equation is a hyperboloid of 1 sheet. The axis is the x axis.
c) This equation is a hyperboloid of 2 sheets. The axis is the x axis.

17. Make a rough sketch of the graph of

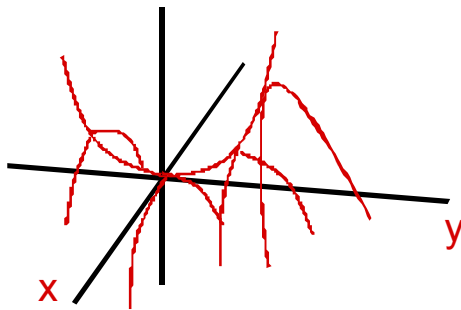
$$\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{4} = 1$$

- Ans: The graph is an ellipsoid with intercepts $(\pm 4, 0, 0)$, $(0, \pm 5, 0)$, and $(0, 0, \pm 2)$.



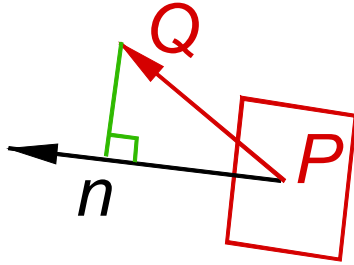
18. Make a sketch of the graph of the surface $z = 2y^2 - x^2$.

- Ans: The graph is a hyperbolic paraboloid.



19. Find the distance between the point $(2, -1, -1)$ and the plane $x - y + z = 4$.

Ans: Get one point on the plane say $(4, 0, 0)$. The vector from $(4, 0, 0)$ to $(2, -1, -1)$ is $(-2, -1, -1)$. The distance is equal to the absolute value of the projection of this vector onto the normal vector $(1, -1, 1)$ to the plane.



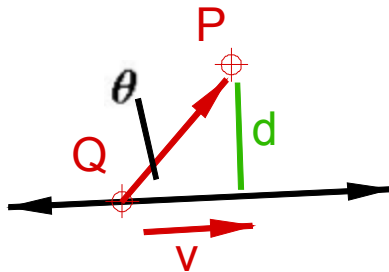
The projection is given by $\|PQ\| \cos \theta$ where

$$\cos \theta = \frac{PQ \cdot n}{\|PQ\| \|n\|} = \frac{(-2, -1, -1) \cdot (1, -1, 1)}{\sqrt{6} \sqrt{3}} = \frac{-2}{3\sqrt{2}}$$

The distance is $\sqrt{6} \times \frac{2}{3\sqrt{2}} = \frac{2}{3} \sqrt{3}$.

20. Find the distance from the point $(3, -8, 1)$ to the line $\frac{x-3}{3} = \frac{y+7}{-1} = \frac{z+2}{5}$.

Ans: A point on the line is $(3, -7, -2)$. We need a vector from this point to $(3, -8, 1)$ which is $PQ = (0, -1, 3)$. A vector parallel to the line is $v = (3, -1, 5)$.



$$d = \|QP\| |\sin \theta| = \frac{\|v\| \|QP\| |\sin \theta|}{\|v\|} = \frac{\|v \times QP\|}{\|v\|}.$$

$$v \times QP = \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 0 & -1 & 3 \end{vmatrix} = 2i - 9j - 3k \quad .$$

$$d = \frac{\sqrt{4 + 81 + 9}}{\sqrt{9 + 1 + 25}} = \frac{\sqrt{94}}{\sqrt{35}} \approx 1.64.$$

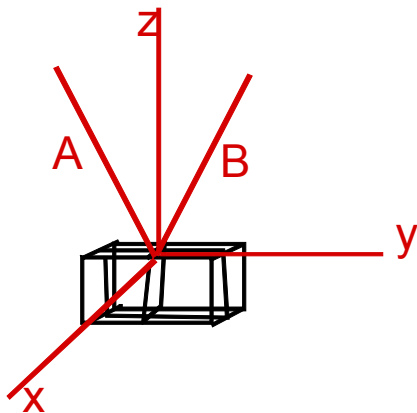
Here is another way to solve this problem where we find θ .

$$\cos \theta = \frac{(0, -1, 3) \cdot (3, -1, 5)}{\sqrt{1 + 9} \sqrt{9 + 1 + 25}} = 0.85524 \Rightarrow$$

$$\theta = 31.21419^\circ.$$

Therefore $d = \|PQ\| \sin \theta = \sqrt{10} \sin 31.21419^\circ = 1.6388$.

21. In the figure below, two ropes are attached to a 500 pound crate. Rope A exerts a force of $(10, -130, 200)$ pounds on the crate and rope B exerts a force of $(-20, 180, 160)$ pounds on the crate. If no further ropes are added, find net force on the crate and the direction it will move. If a 3rd rope C is added to balance the crate, what force must this rope exert on the crate?



Ans: The sum of the forces is

$$(10, -130, 200) + (-20, 180, 160) + (0, 0, -500) = (-10, 50, -140).$$

This has a magnitude of

$$\sqrt{10^2 + 50^2 + 140^2} = 148.997 \blacksquare$$

We can say the direction is

$$\frac{1}{10}(-10, 50, -140) = (-1, 5, -14).$$

If a 3rd rope is to balance the crate then the sum of the forces must be 0. This means the 3rd rope must exert a force of $(10, -50, 140)$.