**Using Substitution**

**Homogeneous and Bernoulli Equations**

Sometimes differential equations may not appear to be in a solvable form. However, if we make an appropriate substitution, often the equations can be forced into forms which we can solve, much like the use of *u* substitution for integration. We must be careful to make the appropriate substitution. Two particular forms of equations lend themselves naturally to substitution.

**Homogeneous Equations** A function *F*(*x,y*) is said to be *homogeneous* if for some *t* 6= 0

*F*(*tx,ty*) = *F*(*x,y*)*.*

That is to say that a function is homogeneous if replacing the variables by a scalar multiple does not change the equation. Please note that the term homogeneous is used for two different concepts in differential equations.

Examples

1.  is homogeneous since



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We say that a differential equation is homogeneous if it is of the form ) for a homogeneous function *F*(*x,y*). If this is the case, then we can make the substitution *y* = *ux*. After using this substitution, the equation can be solved as a *seperable* differential equation. After solving, we again use the substitution *y* = *ux* to express the answer as a function of *x* and *y*.

Example

1. 

We have already seen that the function above is homogeneous from the previous examples. As a result, this is a homogeneous differential equation. We will substitute *y* = *ux*. By the product rule, . Making these substitutions we obtain



Now this equation must be separated.



Integrating this we get,

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Finally we use thatto get our implicit solution .

**Bernoulli Equations** We say that a differential equation is a *Bernoulli Equation* if it takes one of the forms

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These differential equations almost match the form required to be linear. By making a substitution, both of these types of equations can be made to be linear. Those of the first type require the substitution *v* = *ym*+1. Those of the second type require the substitution *u* = *y*1−*n*. Once these substitutions are made, the equation will be *linear* and may be solved accordingly.

Example



You can see that this is a Bernoulli equation of the second form. We make the substitution *u* = *y*1−4 = *y*−3. This gives . The equation will be easier to manipulate if we multiply both sides by *y*−4. Our new equation will be.

Making the appropriate substitutions this becomes .

If we multiply by −3 we see that the equation is now linear in *u* and can be solved:



After undoing the *u* substitution, we have the solution

