Seperable Equations A differential equation is called seperable if it is of the form

\[ g(y)y' = f(x) \]

An equation is seperable if we can isolate all \( y \) terms on one side of the equation and all \( x \) terms on the other side. Equations of this type can be solved by integrating each side of the equation with respect to the appropriate variable.

**Examples**

1. \( y' = yx \)

   This equation is separable, as can be seen after dividing by \( y \). This gives \( \frac{y'}{y} = x \). Integrating both sides gives \( \ln y = x + C \implies y = e^{x+C} = Ce^x \). When we divided by \( y \), we tacitly assumed that \( y \neq 0 \). We must therefore check if \( y = 0 \) solves the differential equation. The solutions are then \( y = 0 \) and \( y = Ce^x \).

2. \( 2xy^2 - x^4y' = 0 \)

   We can rearrange this equation to give \( \frac{2}{x^2} = \frac{y'}{y^2} \). This is separable, and the solution is revealed by integrating. \( \frac{1}{x^2} + C = \frac{1}{y} \implies y = \frac{x^2}{1+Cx^2} \).

**First Order Linear Equations** These differential equations take the general form

\[ y' + p(x)y = q(x) \]

where \( p(x) \) and \( q(x) \) are functions of \( x \) only. The following are examples of linear equations.

1. \( y' + x^2y = 0 \)
2. \( y' + \cos(x) \ y = x^2 \)
3. \( y' + \frac{y}{1-x} = e^x \)

The following equations would not qualify as linear.

1. \( (y')^2 - \sin(x) \ y = 0 \)
2. \( y' + \frac{x^2}{y} = 2x \)
3. \( y' + e^x y = y^2 \)
To solve these equations, we use the integrating factor \( \mu = e^{\int p(x) \, dx} \). With this integrating factor, the solution can then be written as \( y = \frac{1}{\mu} \int \mu \, q(x) \, dx \).

Examples

1. \( y' + \frac{y}{x} = 2e^{x^2} \)

   In this case, \( p(x) = \frac{1}{x} \) and \( \mu = e^{\int \frac{1}{x} \, dx} = e^{\ln x} = x \). Using our above equation for \( y \) gives the solution \( y = \frac{1}{x} \int 2xe^{x^2} \, dx = \frac{1}{x}(e^{x^2} + C) \)

2. \( y' + y \cos x = \cos x \)

   In this case, \( p(x) = \cos x \) and \( \mu = e^{\int \cos x \, dx} = e^{\sin x} \). Again, applying the solution equation gives \( y = \frac{1}{\sin x} \int \cos x \, e^{\sin x} \, dx = e^{-\sin x}(e^{\sin x} + C) = 1 + Ce^{-\sin x} \)

Exact Equations

An equation of the form

\[ M \, dx + N \, dy = 0 \]

with \( M \) and \( N \) functions of \( x \) and \( y \), is said to be exact if \( \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \).

To solve an exact equation, we follow these steps:

1. Our solution will be \( F(x, y) = \Psi(y) + \int M \, dx = C \), where \( \Psi(y) \) is a function entirely of \( y \) to be found later.

2. Calculate the integral \( \int M \, dx \).

3. Take the derivative of \( F(x, y) \) with respect to \( y \). Set this equal to \( N \) and solve for \( \Psi'(y) \).

   \[ \Psi'(y) = N - \frac{\partial \int M \, dx}{\partial y} \]

4. Find \( \Psi(y) \) by integrating \( \Psi'(y) \) with respect to \( y \). \( \Psi(y) = \int \Psi'(y) \, dy \).

5. Plug \( \Psi(y) \) into \( F(x, y) \) to obtain the solution.

Examples

1. \( 2xy \, dx + (x^2 + 2y) \, dy = 0 \)

   Here \( M = 2xy \) and \( N = x^2 + 2y \). We see the equation is exact since \( \frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x} \).

   \[ F(x, y) = \int 2xy \, dx + \Psi(y) = x^2y + \Psi(y) \]

   Now we solve for \( \Psi(y) \).

   \[ \Psi'(y) = N - \frac{\partial(x^2y)}{\partial y} = (x^2 + 2y) - x^2 \implies \Psi'(y) = 2y \]

   Integrating we see that \( \Psi(y) = y^2 \). Our solution is then \( x^2y + y^2 = c \).
2. \((2xy - 9x^2) \, dx + (2y + x^2 + 1) \, dy = 0\)

Here \(M = 2xy - 9x^2\) and \(N = 2y + x^2 + 1\). We see the equation is exact since \(\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}\). 

\[F(x, y) = \int 2xy - 9x^2 \, dx + \Psi(y) = x^2y - 3x^3 + \Psi(y).\]

Next, solve for \(\Psi(y)\).

\[\Psi'(y) = N - \frac{\partial (x^2y - 3x^3)}{\partial y} = (2y + x^2 + 1) - x^2 = 2y + 1.\]

Integrate this to see that \(\Psi(y) = y^2 + y\).

The solution is then \(F(x, y) = x^2y - 3x^3 + y^2 + y = C\).

**Making Equations Exact**

Occasionally, one will encounter an equation of the form

\[M \, dx + N \, dy = 0\]

that does not meet the criterion for exactness. In certain situations, we can find an appropriate integrating factor which will transform this into an exact equation.

**Case 1** Integrating factors of \(x\) only: If the quantity \(p(x) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\) is a function with no occurrences of \(y\), then \(\mu = e^{\int p(x) \, dx}\) is an integrating factor for the differential equation.

**Case 2** Integrating factors of \(y\) only: If the quantity \(p(y) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\) is a function with no occurrences of \(x\), then \(\mu = e^{\int p(y) \, dy}\) is an integrating factor for the differential equation.

When the integrating factor \(\mu\) exists, one may multiply the differential equation by \(\mu\) to created an exact equation.

**Examples**

1. \((y^2(x^2 + 1) + xy) \, dx + (2xy + 1) \, dy = 0\)

   Here \(\frac{\partial M}{\partial y} = 2y(x^2 + 1) + x\) and \(\frac{\partial N}{\partial x} = 2y\). As we can see, this equation is not exact. We will search for an integrating factor. \(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \frac{2y(x^2 + 1) + x - 2y}{2y + 1} = \frac{2yx^2 + x}{2y + 1} = x\). This a function entirely of \(x\) so that \(\mu = e^{\int x \, dx} = e^{\frac{x^2}{2}}\) will be an integrating factor.

   Multiply the initial equation by \(\mu\) to give \((e^{\frac{x^2}{2}}y^2(x^2 + 1) + e^{\frac{x^2}{2}}xy) \, dx + (2e^{\frac{x^2}{2}}xy + e^{\frac{x^2}{2}}) \, dy = 0\).

2. \((x^2y + 2y^2 \sin x) \, dx + (\frac{2}{3}x^3 - 6y \cos x) \, dy = 0\)

   The equation is not exact since \(\frac{\partial M}{\partial y} = x^2 + 4y \sin x\) and \(\frac{\partial N}{\partial x} = 2x^3 + 6y \sin x\). Now attempt to find an integrating factor.

   \[\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x^2 + 6y \sin x - x^2 - 4y \sin x}{x^2y + 2y^2 \sin x} = \frac{x^2 + 2y \sin x}{x^2y + 2y^2 \sin x} = \frac{1}{y}.
\]
This is a function entirely of $y$ so the equation has an integrating factor of the form $e^{\int \frac{1}{y} \, dy} = e^{\ln y} = y$.

Multiply the initial equation by $y$ to give $(x^2y^2 + 2y^3\sin x) \, dx + \left(\frac{2}{3}x^3y - 6y^2\cos x\right) \, dy = 0$. Now $\frac{\partial M}{\partial y} = 2x^2y + 6y^2\sin x = \frac{\partial N}{\partial x}$. As we can see, this equation is now exact and can be solved accordingly.