**First Order Differential Equations**

**Seperable Equations** A differential equation is called seperable if it is of the form

*g*(*y*)*y*0 = *f*(*x*)

An equation is seperable if we can isolate all *y* terms on one side of the equation and all *x* terms on the other side. Equations of this type can be solved by integrating each side of the equation with respect to the appropriate variable.

Examples

1. *y*0 = *yx*

This equation is seperable, as can be seen after dividing by *y*. This gives . Integrating both sides gives ln*y* = *x* + *C* =⇒ *y* = *ex*+*C* = *Cex*. When we divided by *y*, we tacitly assumed that *y* 6= 0. We must therefore check if *y* = 0 solves the differential equation. The soltuions are then *y* = 0 and *y* = *Cex*.

1. 2*xy*2 − *x*4*y*0 = 0

We can rearrange this equation to give . This is seperable, and the solution is revealed by integrating. *.*

**First Order Linear Equations** These differential equations take the general form

*y*0 + *p*(*x*)*y* = *q*(*x*)

where *p*(*x*) and *q*(*x*) are functions of *x* only. The following are examples of linear equations.

1.

2.

3.

The following equations would not qualify as linear.

1.

2.

3.

To solve these equations, we use the integrating factor *µ* = *e*R *p*(*x*) *dx*. With this integrating factor, the solution can then be written as .

Examples

1. 

In this case, and . Using our above equation for *y* gives the solution

2. *y*0 + *y* cos*x* = cos*x*

In this case, *p*(*x*) = cos*x* and *µ* = *e*R cos*x dx* = *e*sin*x*. Again, applying the solution equation gives

**Exact Equations** An equation of the form

*M dx* + *N dy* = 0

with *M* and *N* functions of *x* and *y*, is said to be exact if *∂M∂y* = *∂N∂x* .

To solve an exact equation, we follow these steps:

1. Our solution will be *F*(*x,y*) = Ψ(*y*) + R *M dx* = *C*, where Ψ(*y*) is a function entirely of *y* to be found later.
2. Calculate the integral R *M dx*.
3. Take the derivative of *F*(*x,y*) with respect to *y*. Set this equal to *N* and solve for Ψ0(*y*).

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1. Find Ψ(*y*) by integrating Ψ0(*y*) with respect to *y*. Ψ(*y*) = R Ψ0(*y*) *dy*.
2. Plug Ψ(*y*) into *F*(*x,y*) to obtain the solution.

Examples

1. 2*xy dx* + (*x*2 + 2*y*) *dy* = 0

 Here *M* = 2*xy* and *N* = *x*2 + 2*y*. We see the equation is exact since *∂M∂y* = 2*x* = *∂N∂x* .

*F*(*x,y*) = R 2*xy dx* + Ψ(*y*) = *x*2*y* + Ψ(*y*). Now we solve for Ψ(

(*x*2 + 2*y*) − *x*2 =⇒ Ψ0(*y*) = 2*y*. Integrating we see that Ψ(*y*) = *y*2. Our solution is then *x*2*y* + *y*2 = *c*.

1. (2*xy* − 9*x*2) *dx* + (2*y* + *x*2 + 1) *dy* = 0

 Here *M* = 2*xy* − 9*x*2 and *N* = 2*y* + *x*2 + 1. We see the equation is exact since *∂M∂y* =

). Next, solve for Ψ(*y*).

+1. Integrate this to see that Ψ(*y*) = *y*2 +*y*.

The solution is then *F*(*x,y*) = *x*2*y* − 3*x*3 + *y*2 + *y* = *C*.

**Making Equations Exact** Ocassionally, one will encounter an equation of the form

*M dx* + *N dy* = 0

that does not meet the criterion for exactness. In certain situations, we can find an appropriate

integrating factor which will transform this into an exact equation.

Case 1 Integrating factors of *x* only: If the quantity is a function with no occurances of *y*, then *µ* = *e*R *p*(*x*) *dx* is an integrating factor for the differential equation.

Case 2 Integrating factors of *y* only: If the quantity is a function with no occurances of *x*, then *µ* = *e*R *p*(*y*) *dy* is an integrating factor for the differential equation.

When the integrating factor *µ* exists, one may multiply the differential equation by *µ* to created an exact equation.

Examples

1. (*y*2(*x*2 + 1) + *xy*) *dx* + (2*xy* + 1) *dy* = 0

, and . As we can see, this equation is not exact. We will search

for an integrating factor. . This a function entirely of *x* so that will be an integrating factor.

 2 2 2 2

 Multiply the initial equation by *µ* to give (

Now  so that the equation is now exact and can be solved via the methods previously discussed.

2. 

The equation is not exact since , and . Now attempt to

find an integrating factor. .

This is a function entirely of *y* so the equation has an integrating factor of the form *e*R *y*1 *dy* = *e*ln*y* = *y*.

Multiply the initial equation by *y* to give (

Now *∂M∂y* = 2*x*2*y*+6*y*2 sin*x* = *∂N∂x* . As we can see, this equation is now exact and can be solved accordingly.