Using Substitution
Homogeneous and Bernoulli Equations

Sometimes differential equations may not appear to be in a solvable form. However, if we make an appropriate substitution, often the equations can be forced into forms which we can solve, much like the use of *u* substitution for integration. We must be careful to make the appropriate substitution. Two particular forms of equations lend themselves naturally to substitution.

Homogeneous Equations A function *F*(*x,y*) is said to be *homogeneous* if for some *t* 6= 0

$$F(tx,ty) = F(x,y).$$

That is to say that a function is homogeneous if replacing the variables by a scalar multiple does not change the equation. Please note that the term homogeneous is used for two different concepts in differential equations.

Examples

1. $F(x,y) = \dfrac{-3y}{3x-7y}$ is homogeneous since
$$F(tx,ty) = \frac{-3ty}{3tx-7ty} = \frac{-3ty}{t(3x-7y)} = \frac{-3y}{3x-7y} = F(x,y).$$

2. $F(x,y) = \dfrac{xy^2 + x^3\cos(\frac{2x}{3y})}{5y^2x+y^3}$ is homogeneous since
$$F(tx,ty) = \frac{tx(ty)^2 + (tx)^3\cos(\frac{2tx}{3ty})}{5(ty)^2tx+(ty)^3} = \frac{t^3xy^2 + t^3x^3\cos(\frac{2tx}{3ty})}{5t^2y^2tx+t^3y^3} = \frac{t^3(xy^2 + x^3\cos(\frac{2tx}{3ty}))}{t^3(5y^2x+y^3)}$$
$$= \frac{xy^2 + x^3\cos(\frac{2x}{3y})}{5y^2x+y^3} = F(x,y)$$

We say that a differential equation is homogeneous if it is of the form $\frac{dy}{dx} = F(x,y)$ for a homogeneous function *F*(*x,y*). If this is the case, then we can make the substitution *y* = *ux*. After using this substitution, the equation can be solved as a *seperable* differential equation. After solving, we again use the substitution *y* = *ux* to express the answer as a function of *x* and *y*.

Example

1. $\frac{dy}{dx} = \frac{-3y}{3x-7y}$

We have already seen that the function above is homogeneous from the previous examples. As a result, this is a homogeneous differential equation. We will substitute *y* = *ux*. By the product rule, $\frac{dy}{dx} = x\frac{du}{dx} + u$. Making these substitutions we obtain

$$x\frac{du}{dx} + u = \frac{-3ux}{3x-7ux}$$

Now this equation must be separated.

$$x\frac{du}{dx} + u = \frac{-3ux}{x(3-7u)} \implies x\frac{du}{dx} + u = \frac{-3u}{3-7u} \implies x\frac{du}{dx} = \frac{-6u+7u^2}{3-7u}$$

$$\implies \frac{3-7u}{-6u+7u^2}du = \frac{dx}{x}$$

Integrating this we get,

$$-\frac{1}{2}\ln(-6u+7u^2) = \ln(x) + C \implies \frac{1}{\sqrt{-6u+7u^2}} = cx \implies \frac{1}{-6u+7u^2} = cx^2.$$

Finally we use that $u = \frac{y}{x}$ to get our implicit solution $\frac{x^2}{-6yx+7y^2} = cx^2 \implies -6yx+7y^2 = c$.

Bernoulli Equations We say that a differential equation is a *Bernoulli Equation* if it takes one of the forms

$$y^m\frac{dy}{dx} + p(x)y^{m+1} = q(x)\ , \quad \frac{dy}{dx} + p(x)y = q(x)y^n.$$

These differential equations almost match the form required to be linear. By making a substitution, both of these types of equations can be made to be linear. Those of the first type require the substitution $v = y^{m+1}$. Those of the second type require the substitution $u = y^{1-n}$. Once these substitutions are made, the equation will be *linear* and may be solved accordingly.

Example

(a) $\frac{dy}{dx} - \frac{y}{3x} = e^x y^4$

You can see that this is a Bernoulli equation of the second form. We make the substitution $u = y^{1-4} = y^{-3}$. This gives $\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$. The equation will be easier to manipulate if we multiply both sides by y^{-4}. Our new equation will be $y^{-4}\frac{dy}{dx} - \frac{y^{-3}}{3x} = e^x$.

Making the appropriate substitutions this becomes $\frac{-1}{3}\frac{du}{dx} - \frac{u}{3x} = e^x$.

If we multiply by −3 we see that the equation is now linear in u and can be solved:

$$\frac{du}{dx} + \frac{u}{x} = -3e^x \implies ux = \int -3xe^x dx \implies ux = -3(xe^x - e^x + C)$$

$$\implies u = -3(e^x - \frac{e^x}{x} + \frac{C}{x})$$

After undoing the u substitution, we have the solution

$$\frac{1}{y^3} = -3(e^x - \frac{e^x}{x} + \frac{C}{x}) \implies y^3 = \frac{x}{e^x - xe^x + C}$$